Singularities of hypoelliptic Green functions

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Abstract. This paper is devoted to a precise description of the singularity near the diagonal of the Green function associated to a hypoelliptic operator using a probabilistic approach. Examples and some applications to potential theory are given.

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1. Introduction

Let $X_1, \ldots, X_m$ be smooth vector fields on $\mathbb{R}^d$, $d \geq 3$ such that the Lie algebra generated by $X_1, \ldots, X_m$ is of full rank at every point:

$$\forall x \in \mathbb{R}^d \,,\, \dim \operatorname{Lie}(X_1, \ldots, X_m)(x) = d \,. \tag{1.1}$$

We are interested on the behaviour of the Green function G of the hypoelliptic operator

$$L = \frac{1}{2} \sum_{j=1}^{m} X_j^2 \tag{1.2}$$

on a smooth bounded domain Ω of $\mathbb{R}^d$. G is smooth off the diagonal and we give in this paper a precise description of its singularity near the diagonal.

From the work of Nagel, Stein and Wainger [19] or Sánchez-Calle [20], it is known that the Green function can be estimated in terms of the natural sub-Riemannian distance ρ:

$$|G(x,y)| \leq c \, \frac{\rho(x,y)^2}{\operatorname{vol}(B_\rho(x, \rho(x,y)))} \,.$$

To state a more precise form of these upper bound, let us introduce some notations. For a multi-index $J = (j_1, \ldots, j_p) \in \{1, \ldots, m\}^p$, we shall write $|J| = p$, and

$$X^J = [X_{j_1}, [X_{j_2}, \ldots, [X_{j_{p-1}}, X_{j_p}] \ldots]]$$

will denote the Lie bracket of the vector fields $X_{j_1}, \ldots, X_{j_p}$. For any $k \in \mathbb{N}^*$ and any $x \in \mathbb{R}^d$, we consider

$$C_k(x) = \text{Span}\,\{X^J(x)\,,\,|J| \le k\}$$

and

$$r(x) = \inf\{k \,:\, \dim C_k(x) = d\}\,. \tag{1.3}$$

By (1.1), $r(x)$ is finite.

Let us denote by $Q(x)$ the graded dimension at x:

$$Q(x) = \sum_{k=1}^{r(x)} k\,(\dim C_k(x) - \dim C_{k-1}(x))\,. \tag{1.4}$$

We shall assume that the geometry of the brackets is locally constant near x, that is, for every $k \in \mathbb{N}^*$ and every y in a neighbourhood $A(x)$ of x, $\dim C_k(y) = \dim C_k(x)$. Then, of course, $r(y)$ and $Q(y)$ are constant on this neighbourhood. Since we want to exclude the trivial elliptic cases where $d = Q = 2$ and $d = Q = 3$, we assume $Q \ge 4$.

Following [2], we shall introduce a useful coordinate chart. For a fixed $x \in \Omega$ we choose a family of multi-indices $B = \{J_1, \ldots, J_d\}$, such that $\{X^J(x) \,:\, J \in B\}$ is a triangular basis. That is, for every $k \le r$, $\{X^J(x) \,:\, J \in B\,, |J| \le k\}$ generates $C_k(x)$. We shall denote the length $|J_j| = l_j$, $j = 1, \ldots, d$. There exists a neighbourhood W of 0 such that the mapping

$$u \mapsto \varphi_x(u) = \exp\left(\sum_{j=1}^{d} u_j X^{J_j}\right)(x) \tag{1.5}$$

defines a diffeomorphism of W on $\varphi_x(W)$. There exists a neighbourhood U of x such that $U \subset \varphi_x(W) \cap A(x)$.

For $y \in U$, $y = \varphi_x(u)$ we shall denote

$$|y|_x = \left[\sum_{k=1}^{r}\left(\sum_{j, l_j = k} u_j^2\right)^{\frac{Q}{2k}}\right]^{\frac{1}{Q}} \tag{1.6}$$

and we shall show that the estimate of Nagel, Stein and Wainger [19], can be written as

(1.7) $$|G(x,y)| \le \frac{c}{|y|_x^{Q(x)-2}}\,.$$

We want to give a sharper description of the singularity of $G(x,y)$ when $y \to x$. For this purpose we introduce the homogeneous angular variable, for $y \in U \setminus \{x\}$, $y = \varphi_x(u)$,

(1.8) $$\theta_x(y) = \left(\frac{u_1}{|y|_x^{l_1}}, \ldots, \frac{u_d}{|y|_x^{l_d}}\right)\,.$$

Then our main result will be:

(1.9) THEOREM. *There exists a smooth function $\Phi_x > 0$ such that,*

(1.10) $$\lim_{\varepsilon\downarrow 0}\ \sup_{\|x-y\|<\varepsilon}\left|G(x,y)\,|y|_x^{Q(x)-2} - \Phi_x(\theta_x(y))\right| = 0\,.$$

Here and elsewhere $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^d$.

This new geometric coefficient Φ will be described in §5 as the density of the occupation measure for a process (u_t), that we call tangent process. This process, non-markovian in general, will be seen as a projection of a left invariant diffusion process on a free nilpotent Lie group.

It must be noticed that, in general, computing Φ is not easy. The value of Φ is computable in some examples (see §9), for instance on Heisenberg groups.

Theorem (1.9) shows that, in general, the limit

$$\lim_{y\to x} G(x,y)\,|y|_x^{Q(x)-2}$$

does not exist; it exists only "radially", that is, if y approaches x in such a way that the angular variable $\theta_x(y)$ tends to a limit. This is in contrast with the elliptic situation, the Heisenberg group situation or the "curved" Heisenberg group situation studied by Chaleyat-Maurel and Le Gall [9], where Φ_x is constant.

Our approach for the proof of the Theorem (1.9) is probabilistic. It relies on results on stochastic Taylor expansion of paths of the diffusion generated by L and on the a priori estimate given by Nagel, Stein and Wainger [19]. We follow and extend the strategy given by Chaleyat-Maurel and Le Gall [9] in a simple context.

One must also notice that the behaviour of the heat kernel $p_t^\Omega(x,y)$ on the diagonal has been studied using the same probabilistic tools in [2] or Léandre [18]. The results can be compared with the Theorem (1.9):

(1.11) $$p_t^\Omega(x,x) \sim \frac{c_0(x)}{\sqrt{t}^{\,Q(x)}}\,,$$

where $c_0(x)$ is the density of the law of the tangent process (u_t) taken at time 1.

An example of different behaviour near the diagonal of the Green function in presence of a drift was studied in [14]. The result could be compared with the results in [3] and give an idea of the pathologies for the behaviour which could appear in presence of a drift.

The plan of the paper is as follows: in §2 we introduce the stochastic Taylor expansion and the tangent process, which we study in §3. In §4 − 6 we prove the Theorem (1.9) except for some technical lemmas postponed to the Appendix. We then apply our results to some potential theoretical problems in §7 − 8: estimates of the capacities of small sets, of the volume of the Wiener sausage of small radius, double points. In §9 we give examples where direct computations illustrate our general theorem and even two examples where the conclusion of the theorem is valid though our hypothesis of locally constant geometry fail.

2. Taylor stochastic expansion

Let $(B^1, \dots, B^m)$ be a m-dimensional Brownian motion and consider (x_t) the solution of the Stratonovich equation

$$dx_t = \sum_{j=1}^{m} X_j(x_t) \circ dB_t^j \,,\; x_0 = x \,, \tag{2.1}$$

killed at the first exit time from Ω, $\tau = \inf\{t > 0 \,,\, x_t \notin \Omega\}$. It is known that $\tau < \infty$, P_x-a.s., for every $x \in \mathbb{R}^d$.

By hypoellipticity, for every $x \in \Omega$, $t > 0$, the law of x_t under P_x has, on Ω, a density with respect to the Lebesgue measure, $p_t^\Omega(x, y)$. It is the heat kernel associated to L on Ω and the Green function is

$$G(x, y) = \int_0^\infty p_t^\Omega(x, y)dt \,,\; x, y \in \Omega \,. \tag{2.2}$$

G is the density of occupation measure of (x_t), that is, for every positive measurable function f,

$$E_x \int_0^\tau f(x_t)dt = \int_\Omega f(y)G(x, y)dy \,. \tag{2.3}$$

(2.4) REMARK. We note that for the study of the singularity of G near the diagonal it suffices to consider Ω as a bounded neighbourhood of x, $\Omega \subset U$. Indeed, if we denote by G_V the Green function of L on a neighbourhood V of x, then the singular behaviour near the diagonal of G and G_V is the same, because $L(G - G_V) = 0$. From now on we shall assume that $\Omega \subset U$.

We are interested in the study of the process in short time, $(x_{\varepsilon^2 t})$, $\varepsilon > 0$.

For every $x \in \Omega$, it has the same law under P_x as the solution of the equation

$$dx_t^\varepsilon = \varepsilon \sum_{j=1}^{m} X_j(x_t^\varepsilon) \circ dB_t^j \,,\; x_0^\varepsilon = x \,, \tag{2.5}$$

killed at the first exit time from Ω, $\tau_\varepsilon = \tau/\varepsilon^2$.

Let us consider, for $\lambda > 0$, the dilation defined on $\mathbb{R}^d$,

$$T_\lambda(u) = (\lambda^{l_1} u_1, \ldots, \lambda^{l_d} u_d) \,. \tag{2.6}$$

For $0 \le t < \tau_\varepsilon$, we define the diffusion $(v_t^{(\varepsilon,x)})$, starting from 0,

$$v_t^{(\varepsilon,x)} = (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x_t^\varepsilon) \,. \tag{2.7}$$

We shall introduce a new process, called tangent process. For a multi-index $J = (j_1, \ldots, j_p)$, we denote by B_t^J the Stratonovich iterated integral

$$B_t^J = \int_{0<t_1<\ldots<t_p<t} dB_{t_1}^{j_1} \circ \ldots \circ dB_{t_p}^{j_p} \tag{2.8}$$

and by c_t^J the completely explicit linear combination of Stratonovich iterated integrals

$$c_t^J = \sum_{\tau \in \sigma_{|J|}} \frac{(-1)^{e(\tau)}}{|J|^2 \begin{pmatrix} |J|-1 \\ e(\tau) \end{pmatrix}} B_t^{J\circ\tau^{-1}} \,. \tag{2.9}$$

Here, for a permutation $\tau \in \sigma_p$, of order p, we denoted $e(\tau)$ the number of errors in ordering $\tau(1), \ldots, \tau(p)$ and

$$J \circ \tau = (j_{\tau(1)}, \ldots, j_{\tau(p)}) \,.$$

Recall that $\{X^J(y) : J \in B\}$ is a triangular basis for y close to x. So, for any multi-index L, there exists smooth functions, defined on a neighbourhood of x, $(a_J^L)_{J\in B}$, such that

$$X^L = \sum_{J\in B} a_J^L X^J \,. \tag{2.10}$$

DEFINITION. We shall call *tangent proces*, the process,

$$u_t^{(x)} = \left(\sum_{L,|L|=|J|} a_J^L(x) c_t^L \right)_{J\in B} \,. \tag{2.11}$$

(2.12) PROPOSITION. *Let fix $T > 0$. Then, for any bounded Lipschitz continuous function f on $\mathbb{R}^d$ and for sufficiently small $\varepsilon > 0$, there exists a positive constant c, such that,*

$$\left| E_0 \left(\mathbb{I}_{(T<\tau_\varepsilon)} \int_0^T f(v_t^{(\varepsilon,x)}) dt \right) - E_0 \int_0^T f(u_t^{(x)}) dt \right| \leq c \, \|f\|_{\mathrm{Lip}} \, T \, \varepsilon \, . \tag{2.13}$$

Here we denoted

$$\|f\|_{\mathrm{Lip}} = \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|} \, .$$

To prove this result, we shall use the results of [1] or [8] on the asymptotic expansion in small time of x_t in terms of Lie brackets and iterated Stratonovich integrals. According to the Theorem 4.1 [8], p. 234, for $t \leq T$,

$$x_t^\varepsilon = \exp\left(\sum_{k=1}^r \varepsilon^k \sum_{L, |L| = k} c_t^L X^L \right)(x) + \varepsilon^{r+1} R_{r+1}(\varepsilon, t) \, . \tag{2.14}$$

Here $R_{r+1}(\varepsilon, t)$ is bounded in probability. More precisely, there exists $\alpha, c > 0$ such that, for every $R > c$

$$\lim_{\varepsilon \downarrow 0} P\left(\sup_{0 \leq t \leq T} \|R_{r+1}(\varepsilon, t)\| \geq R \right) \leq \exp\left(-\frac{R^\alpha}{c\,T} \right) \, . \tag{2.15}$$

Proof of the Proposition (2.12). We can write

$$\begin{aligned} &\left| E_0 \left(\mathbb{I}_{(T<\tau_\varepsilon)} \int_0^T f(v_t^{(\varepsilon,x)}) dt \right) - E_0 \int_0^T f(u_t^{(x)}) dt \right| \\ &\leq \left| E_0 \left(\mathbb{I}_{(T<\tau_\varepsilon)} \int_0^T f(v_t^{(\varepsilon,x)}) dt \right) - E_0 \left(\mathbb{I}_{(T<\tau_\varepsilon)} \int_0^T f(u_t^{(x)}) \right) dt \right| \\ &\quad + \|f\|_{\mathrm{Lip}} \, T \, P(T \geq \tau_\varepsilon) \, . \end{aligned}$$

By the classical exponential inequality we know that, there exists two positive constants c, c', such that

$$P(T \geq \tau_\varepsilon) \leq c \, e^{-\frac{c'}{\varepsilon^2 T}} \, .$$

We shall study only the first term.

Let us consider ψ_x the diffeomorphism

$$\psi_x((v_L)_{|L| \leq r}) = \exp\left(\sum_{L, |L| \leq r} v_L X_L \right)(x)$$

and we denote

$$T^{\varepsilon}_{\bar\Omega} = \inf\{t > 0 : (\varepsilon^{|L|} c^L_t)_{|L|\le r} \notin \psi_x^{-1}(\bar\Omega)\}\,. \tag{2.16}$$

We can write,

$$\left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)} \int_0^T f(v_t^{(\varepsilon,x)})dt \right) - E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)} \int_0^T f(u_t^{(x)}) \right) dt \right|$$

$$\le \left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^{\varepsilon}_{\bar\Omega})} \int_0^T f(v_t^{(\varepsilon,x)})dt \right) - E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^{\varepsilon}_{\bar\Omega})} \int_0^T f(u_t^{(x)}) \right) dt \right|$$
$$+2\,\|f\|_{\mathrm{Lip}}\, T\, P(T \ge T^{\varepsilon}_{\bar\Omega})\,.$$

As in [8], p. 238, we have that, for sufficiently small ε,

$$P(T \ge T^{\varepsilon}_{\bar\Omega}) \le \sum_{L,|L|\le r} \exp\left(-\frac{c_L}{\varepsilon^{2|L|}T}\right)\,. \tag{2.17}$$

So, it remains to consider the first term:

$$\left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^{\varepsilon}_{\bar\Omega})} \int_0^T f(v_t^{(\varepsilon,x)})dt \right) - E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^{\varepsilon}_{\bar\Omega})} \int_0^T f(u_t^{(x)}) \right) dt \right|$$

$$\le \|f\|_{\mathrm{Lip}}\, T\, E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^{\varepsilon}_{\bar\Omega})} \sup_{0\le t\le T} \|v_t^{(\varepsilon,x)} - u_t^{(x)}\| \right)\,.$$

Hence, to finish the proof of (2.13), it suffices to prove the following:

(2.18) LEMMA. *There exists a positive constant c, such that for any sufficiently small $\varepsilon > 0$,*

$$E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^{\varepsilon}_{\bar\Omega})} \sup_{0\le t\le T} \|v_t^{(\varepsilon,x)} - u_t^{(x)}\| \right) \le c\,\varepsilon\,. \tag{2.19}$$

Proof. For $J \in B$ and $t < T^{\varepsilon}_{\bar\Omega}$, we denote

$$u_J(\varepsilon, t, x) = (\varphi_x^{-1} \circ \psi_x)_J((\varepsilon^{|L|} c^L_t)_{|L|\le r})\,.$$

We have that, for $J \in B$ and $t < T^{\varepsilon}_{\bar\Omega}$,

$$(\partial_\varepsilon)^k u_J(\varepsilon, t, x)_{|\varepsilon=0} = 0\,, \text{ if } k < |J|\,.$$

Indeed, by the triangularity of the basis $\{X^J(y) : J \in B\}$, for y close to x, we have, for $J \in B$,

$$a^L_J \equiv 0\,, \text{ if } |L| < |J|\,,$$

on a neighbourhood of x. So, for $J \in B$,

$$[\partial(\varphi_x^{-1} \circ \psi_x)_J / \partial v_L]_{|v=0} = a_J^L(x) = 0\,, \text{ if } |L| < |J|\,.$$

Moreover, by the last equality we also have that, for $J \in B$ and $t < T_\Omega^\varepsilon$,

$$(\partial_\varepsilon)^{|J|} u_J(\varepsilon, t, x)_{|\varepsilon=0} = \sum_{L, |L|=|J|} a_J^L(x)(\partial_\varepsilon)^{|L|}(\varepsilon^{|L|} c_t^L)_{|\varepsilon=0}\,,$$

because the terms coresponding to L, with $|L| > |J|$, are zero having the factor $\varepsilon^{|L|}$ (see also [2], pp. 93-94).

Hence, the Taylor expansion around $\varepsilon = 0$ of $u_J(\varepsilon, t, x)$, for $J \in B$ and $t < T_\Omega^\varepsilon$, can be written,

$$u_J(\varepsilon, t, x) = \frac{\varepsilon^{|J|}}{|J|!}(\partial_\varepsilon)^{|J|} u_J(\varepsilon, t, x)_{|\varepsilon=0} + \varepsilon^{|J|+1} R_{J,|J|+1}(\varepsilon, t, x)\,,$$

or

$$\frac{1}{\varepsilon^{|J|}} u_J(\varepsilon, t, x) = \sum_{L, |L|=|J|} c_t^L a_J^L(x) + \varepsilon\, R_{J,|J|+1}(\varepsilon, t, x)\,.$$

Here, for $J \in B$ and $t < T_\Omega^\varepsilon$,

$$R_{J,|J|+1}(\varepsilon, t, x) = \int_0^1 (\partial_\varepsilon)^{|J|+1} u_J(\varepsilon\,\xi, t, x)\frac{(1-\xi)^{|J|}}{|J|!}\,d\xi\,.$$

Using properties (P1), (P2) in [8], p. 238, we see that, for every $J \in B$, there exists $\alpha_J, c_J > 0$, such that, for any $R > c_J$ and for $\varepsilon > 0$ sufficiently small,

$$P\left(\sup_{0 \le t \le T} |R_{J,|J|+1}(\varepsilon, t, x)| \ge R\,;\, T < T_\Omega^\varepsilon\right) \le \exp\left(-\frac{R^{\alpha_J}}{c_J T}\right)\,. \tag{2.20}$$

Indeed, B_t^J satisfies (2.20) and we get the same thing for $u_J(\varepsilon, t, x)$, using its definition in terms of $(\varepsilon^{|L|} c_t^L)$. Then we obtain (2.20).

By (2.14), we have, for $t < T \wedge T_\Omega^\varepsilon$,

$$(T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x_t^\varepsilon - \varepsilon^{r+1} R_{r+1}(\varepsilon, t)) = \left(\sum_{L, |L|=|J|} c_t^L a_J^L(x) + \varepsilon\, R_{J,|J|+1}(\varepsilon, t, x)\right)_{J \in B}\,.$$

We note that, for any $0 < \varepsilon < 1$ and any $u \in \mathbb{R}^d$, $\|T_{\frac{1}{\varepsilon}}(u)\| \le \frac{1}{\varepsilon^r}\|u\|$. Therefore, by the Lipschitz property of φ_x^{-1}, we can write, for $t \le T \wedge \tau_\varepsilon \wedge T_\Omega^\varepsilon$,

$$\|v_t^{(\varepsilon,x)} - u_t^{(x)}\| \le \|v_t^{(\varepsilon,x)} - (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x_t^\varepsilon - \varepsilon^{r+1} R_{r+1}(\varepsilon, t))\|$$

$$+\|(T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x_t^\varepsilon - \varepsilon^{r+1} R_{r+1}(\varepsilon, t)) - u_t^{(x)}\| \le c\,\frac{1}{\varepsilon^r}\,\|\varepsilon^{r+1} R_{r+1}(\varepsilon, t)\|$$

$$+\|\varepsilon\,(R_{J,|J|+1}(\varepsilon,t,x))_{J\in B}\|\,.$$

Hence, for $t\le T\wedge\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega}$,

$$\|v_t^{(\varepsilon,x)}-u_t^{(x)}\|\le\varepsilon\,R(\varepsilon,t)\,, \tag{2.21}$$

where, for $t\le T\wedge\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega}$,

$$R(\varepsilon,t)=c\,\|R_{r+1}(\varepsilon,t)\|+\|(R_{J,|J|+1}(\varepsilon,t,x))_{J\in B}\|\,. \tag{2.22}$$

Using (2.15) and (2.20) we prove the existence of positive constants α',c', such that, for any $R>c'$ and for $\varepsilon>0$ sufficiently small,

$$P\left(\sup_{0\le t\le T}R(\varepsilon,t)\ge R\,;\,T<\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega}\right)\le\exp\left(-\frac{R^{\alpha'}}{c'T}\right)\,. \tag{2.23}$$

Finally, by (2.21), we can write,

$$E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega})}\sup_{0\le t\le T}\|v_t^{(\varepsilon,x)}-u_t^{(x)}\|\right)\le\varepsilon\,E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega})}\sup_{0\le t\le T}R(\varepsilon,t)\right)$$

$$=\varepsilon\int_0^\infty P\left(\sup_{0\le t\le T}R(\varepsilon,t)\ge R\,;\,T<\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega}\right)dR\,.$$

Now, (2.19) follows from this, using (2.23) and the lemma is proved.

This also ends the proof of the Proposition (2.12).

3. Study of the tangent process

The process $(u_t^{(x)})$ is not necessarily a diffusion process. However, we shall prove that it is the image by a projection of a left invariant diffusion on a nilpotent group.

We denote by $g(m,r)$ the free r-nilpotent Lie algebra with m generators $Y_1,\dots,Y_m$. We shall identifie $g(m,r)$ and the associated simple connected nilpotent Lie group $\mathcal{N}(m,r)$, which is nothing but $g(m,r)$ with the multiplication given by the Campbell-Hausdorff formula. We denote, by a clear abuse of notation, Y_j the left invariant vector field on $\mathcal{N}(m,r)$ defined by the generator Y_j of $g(m,r)$.

Let us consider $(\mathcal{G}_t)$ the invariant diffusion on $\mathcal{N}(m,r)$. That is the solution, starting from the unit element, $e\in\mathcal{N}(m,r)$, of the Stratonovich equation

$$d\mathcal{G}_t=\sum_{j=1}^m Y_j(\mathcal{G}_t)\circ dB_t^j\,,\ \mathcal{G}_0=e\,. \tag{3.1}$$

(3.2) PROPOSITION. *There exists a unique linear projection, π_x, such that*

$$
(3.3) \qquad u_t^{(x)} = \pi_x(\mathcal{G}_t)\,.
$$

Proof. According to the result of the Proposition 3.1 [8], p. 228,

$$
\mathcal{G}_t = \exp\left(\sum_{L,|L|\le r} c_t^L Y^L\right)(e)\,.
$$

Let $\{Y^K : K \in A\}$ be a Hall basis of $g(m,r)$. Then, for every multi-index L,

$$
(3.4) \qquad Y^L = \sum_{K\in A,|K|=|L|} c_K^L Y^K\,,
$$

with universal constants c_K^L. Let us denote, for $K \in A$,

$$
b_t^K = \sum_{L,|L|=|K|} c_K^L c_t^L\,.
$$

and then, by a simple calculation, we get that

$$
\mathcal{G}_t = \exp\left(\sum_{K\in A} b_t^K Y^K\right)(e)\,.
$$

We note that, by the properties of vector fields, (3.4) it is also true with X_j instead Y_j. By the fact that $\{X^J(x) : J \in B\}$ is a basis, we see that $u_t^{(x)}$ can be written:

$$
u_t^{(x)} = \left(\sum_{K\in A,|K|=|J|} a_J^K(x) b_t^K\right)_{J\in B}\,.
$$

Put $n = \dim g(m,r) - d$ and $A = \{K_i : i = 1,\dots,d+n\}$. There exists a diffeomorphism between $\mathbb{R}^{d+n}$, and $\mathcal{N}(m,r)$,

$$
w \mapsto \phi_e(w) = \exp\left(\sum_{i=1}^{d+n} w_i Y^{K_i}\right)(e)\,.
$$

Let us denote by $p_x : \mathbb{R}^{d+n} \to \mathbb{R}^d$ the projection

$$
p_x(w) = \left(\sum_{i,|K_i|=|J_j|} a_j^i(x)\, w_i\right)_{j=1,\dots,d} = \tilde{M}(x)\, w\,.
$$

Here we denoted $a_j^i(x) = a_{J_j}^{K_i}(x)$, $j = 1,\dots,d$, $i = 1,\dots,d+n$ and $\tilde{M}(x)$ is the matrix with elements $a_j^i(x)$ if $|J_j = |K_i|$ and zero otherwise.

Hence, taking

(3.5) $$\pi_x = p_x \circ \phi_e^{-1}\,,$$

we obtain (3.3).

(3.6) COROLLARY. *For every $t > 0$, the law of $u_t^{(x)}$ has a smooth density with respect to the Lebesgue measure, $q_t^{(x)}(0,u)$.*

Proof. We show that the Malliavin covariance matrix of $u_t^{(x)}$ is not degenerate for every $t > 0$. It is known that the Malliavin covariance matrix of $\mathcal{G}_t$ is not degenerate for $t > 0$. The same thing is true for $b_t = \phi_e^{-1}(\mathcal{G}_t)$. But, by (3.3),

$$u_t^{(x)} = \tilde{M}(x)\ b_t\,,$$

and we conclude, noting that $\tilde{M}(x)$ is a full rank matrix.

Let us denote, for $u \in \mathbb{R}^d \setminus \{0\}$,

(3.7) $$g^{(x)}(0,u) = \int_0^\infty q_t^{(x)}(0,u)dt\,,$$

the density of the occupation measure of the process $(u_t^{(x)})$. That is, for every positive measurable function f,

(3.8) $$E_0 \int_0^\infty f(u_t^{(x)})dt = \int_{\mathbb{R}^d} g^{(x)}(0,u)f(u)du\,.$$

(3.9) PROPOSITION. *$g^{(x)}(0,\cdot)$ is a strictly positive smooth function on $\mathbb{R}^d \setminus \{0\}$.*

Proof. The fact that $g^{(x)}$ is smooth follows from (3.7). We show now that $g^{(x)}$ is a strictly positive function. We denote by $G^{(N)}$ the Green function of the diffusion $(\mathcal{G}_t)$. Then, for every positive measurable function f,

$$E_e \int_0^\infty f(\mathcal{G}_t)dt = \int_{\mathcal{N}(m,r)} G^{(N)}(e,g)f(g)dg\,,$$

where dg denotes the Haar measure on $\mathcal{N}(m,r)$.

It is known that $G^{(N)}$ is a strictly positive function (see for instance [12], p. 102). Using again (3.3), we shall write $g^{(x)}$ in terms of $G^{(N)}$ as an integral on a fiber of the projection map π_x, and we shall conclude. We prove:

(3.10) $$g^{(x)}(0,u) = c\int_{\mathbb{R}^n} G^{(N)}(\phi_e(0,0), \phi_e(u - M(x)h, h)dh\,.$$

Here $c > 0$ and $M(x)$ is the block of the matrix $\tilde{M}(x)$, having d lines indexed by B and n columns indexed by $A \setminus B$. Indeed, we have

$$\int_{\mathbb{R}^d} g^{(x)}(0,u) f(u) du = E_0 \int_0^\infty f(u_t^{(x)}) dt$$

$$= E_e \int_0^\infty (f \circ \pi_x)(\mathcal{G}_t) dt = \int_{\mathcal{N}(m,r)} G^{(N)}(e,g)(f \circ \pi_x)(g) dg$$

$$= c \int_{\mathbb{R}^d \times \mathbb{R}^n} G^{(N)}(\phi_e(0,0), \phi_e(u,h)(f \circ \pi_x)(\phi_e(u,h) du\, dh\,,$$

where $c > 0$ is the absolute value of the jacobian of ϕ_e. In the latter integral we perform the change of variables $v = u + D(x)h$. Since f was an arbitrary function we get (3.10) and the proposition is proved.

We show now that the time spent by $(u_t^{(x)})$ in a Euclidian ball is finite:

(3.11) PROPOSITION. *For every* $\rho > 0$,

$$E_0 \int_0^\infty 1\!\!1_{B(0,\rho)}(u_t^{(x)}) dt < \infty\,. \tag{3.12}$$

Before proving this result we shall make a useful remark. We note that, in this nilpotent context, the estimate of the Green function (1.7), can be written:

$$|G^{(N)}(e,g)| \le \frac{c}{|g|_N^{Q_N - 2}}\,,\; g \ne e\,, \tag{3.13}$$

where the homogeneous norm of $g = \phi_e(w)$, $w \in \mathbb{R}^{d+n}$, is

$$|g|_N = \left[\sum_{k=1}^{r} \left(\sum_{i,|K_i|=k} w_i^2 \right)^{\frac{Q_N}{2k}} \right]^{\frac{1}{Q_N}}\,. \tag{3.14}$$

Here Q_N is the homogeneous dimension of $\mathcal{N}(m,r)$,

$$Q_N = \sum_{k=1}^{r} k\, \mathrm{dim} V_k\,, \tag{3.15}$$

with

$$V_k = \mathrm{Span}\{Y^J : |J| = k\}\,, k = 1, \ldots, r\,.$$

V_k's form the natural graduation of the Lie algebra, $g(m,r) = V_1 \oplus \ldots \oplus V_r$.

Proof of the Proposition (3.11). By (3.10), we can write

$$E_0 \int_0^\infty 1\!\!1_{B(0,\rho)}(u_t^{(x)}) dt = \int_{B(0,\rho)} g^{(x)}(0,u) f(u) du$$

$$= c \int_{B(0,\rho)} du \int_{\mathbb{R}^n} G^{(N)}(\phi_e(0,0), \phi_e(u - M(x)h, h)dh$$

$$= c \int_{F_x(B(0,\rho)\times\mathbb{R}^n)} G^{(N)}(\phi_e(0,0), \phi_e(v,h)dv\, dh\,,$$

where we denoted $F_x(u,h) = (p_x(u,h), h)$. So, by (3.13),

$$E_0 \int_0^\infty \mathbb{1}_{B(0,\rho)}(u_t^{(x)})dt \le c \int_{\|\pi_x(g)\|<\rho} \frac{dg}{|g|_N^{Q_N-2}}\,.$$

The right hand side of this last inequality is finite (see Lemma (A.7)).

(3.16) COROLLARY. *For every* $\rho > 0$, *for every continuous function* f *on* $\mathbb{R}^d$, *bounded by* 1, *with support in* $B(0,\rho)$, *and for every* $\delta > 0$, *there exists* $T(\delta) > 0$ *such that,*

$$\left| E_0 \int_{T(\delta)}^\infty f(u_t^{(x)})dt \right| \le \delta\,. \tag{3.17}$$

(3.18) COROLLARY. *For* $t > 0$, *we denote by* $\mu_t^{(x)}$ *the law of* $u_t^{(x)}$. *Then, for every* $\rho > 0$ *and for every* $\delta > 0$, *there exists* $T(\delta) > 0$ *such that,*

$$\mu_{T(\delta)}^{(x)}(B(0,\rho)) \le \delta\,. \tag{3.19}$$

Proof. We get the convergence of the integral $\int_0^\infty P_0(u_t^{(x)} \in B(0,\rho))dt$, using (3.12). Hence, $\lim_{t\uparrow\infty} \mu_t^{(x)}(B(0,\rho)) = 0$.

4. Study of the rescaled diffusion

We shall analyse now the diffusion $(v_t^{(\varepsilon,x)})$. We shall prove the following:

(4.1) PROPOSITION. *For every* $0 < \rho < 1$ *and for every continuous function* f *on* $\mathbb{R}^d$, *bounded by* 1, *with support in* $B(0,\rho)$,

$$\lim_{\varepsilon\downarrow 0, T\uparrow\infty} E_0 \left(\mathbb{1}_{(T<\tau_\varepsilon)} \int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt \right) = 0\,. \tag{4.2}$$

Proof. Let $G^{(\varepsilon,x)}$ be the Green function of $(v_t^{(\varepsilon,x)})$. For every positive measurable f,

$$E_0 \int_0^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt = \int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} G^{(\varepsilon,x)}(0,u)f(u)du\,. \tag{4.3}$$

We can write

$$E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt\right) = E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}E_{v_T^{(\varepsilon,x)}}\int_0^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt\right)$$

$$= E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} G^{(\varepsilon,x)}(v_T^{(\varepsilon,x)},u)f(u)du\right)$$

$$= \int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} d\mu_T^{(\varepsilon,x)}(v)\int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} G^{(\varepsilon,x)}(v,u)f(u)du\,.$$

Here $\mu_T^{(\varepsilon,x)}$ denotes the measure having the density $\mathbb{1}_{(T<\tau_\varepsilon)}$ with respect to the law of $v_T^{(\varepsilon,x)}$. We shall estimate the integral of $G^{(\varepsilon,x)}$.

It is a simple calculation to show that, for $v,u\in(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)$,

$$G^{(\varepsilon,x)}(v,u) = \varepsilon^{Q-2}G(v_\varepsilon^x,u_\varepsilon^x)\,, \tag{4.4}$$

Here we denoted $u_\varepsilon^x = (\varphi_x\circ T_\varepsilon)(u)$, for $x\in\Omega$, $\varepsilon>0$ sufficiently small and $u\in\mathbb{R}^d$.

Therefore, by (1.8), we get

$$\int_{B(0,\rho)} G^{(\varepsilon,x)}(v,u)du \le \int_{B(0,\rho)} \frac{c\,\varepsilon^{Q-2}du}{|u_\varepsilon^x|_{v_\varepsilon^x}^{Q-2}}\,, \tag{4.5}$$

for $\varepsilon>0$ sufficiently small and $u\in(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)$.

(4.6) LEMMA. *For any $v\in\mathbb{R}^d$ and for $\varepsilon>0$ sufficiently small, there exists a constant $c>0$, such that*

$$\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}du}{|u_\varepsilon^x|_{v_\varepsilon^x}^{Q-2}} < c\,. \tag{4.7}$$

Moreover,

$$\lim_{\|v\|\uparrow\infty}\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}du}{|u_\varepsilon^x|_{v_\varepsilon^x}^{Q-2}} = 0\,, \tag{4.8}$$

uniformly in $\varepsilon>0$.

Proof. For the first part we write the integral as

$$\varepsilon^{-2}\int_{(\varphi_x\circ T_\varepsilon)(B(0,\rho))}\frac{dy''}{|y''|_{z_\varepsilon^x}^{Q-2}}\,.$$

(4.7) is a particular case of the following estimate: there exists a positive constant c such that, for $\varepsilon>0$ sufficiently small

$$\sup_z \int_{|y|_z<\varepsilon} \frac{dy}{|y|_z^{Q-2}} < c\,\varepsilon^2\,, \tag{4.9}$$

with the supremum taken for z in a neighbourhood of x. We shall now prove (4.9). Firstly, by the change of variables $v = (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(y)$, we get that

$$\int_{|y|_x<\varepsilon} \frac{dy}{|y|_x^{Q-2}} = c\,\varepsilon^2 \int_{|v|_h<1} \frac{dv}{|v|_h^{Q-2}} < c\,\varepsilon^2\,,$$

as follows from the Lemma (A.1) of the Appendix. Here and elsewhere $|u|_h$ denotes the homogeneous norm of $u \in \mathbb{R}^d$:

$$|u|_h = \left[\sum_{k=1}^r \left(\sum_{j,l_j=k} u_j^2 \right)^{\frac{Q}{2k}} \right]^{\frac{1}{Q}} .$$

To get (4.9) it suffices to note that the bound in Lemma (A.1) depends only on the radius of the homogeneous ball (here equal to 1). Since $\{X^{J_j}(z) : j = 1,\ldots,d\}$, is a triangular basis, for z close enough to x, we conclude by a smooth change of coordinates.

In proving (4.8) we use some simple properties of the locally homogeneous norm (see (6.9), (6.11)). There exists some constants $c_0, c', c'' > 0$, such that

$$\sup_{\|u\|<\rho} \frac{\varepsilon^{Q-2}}{|u_\varepsilon^x|_{v_\varepsilon^x}^{Q-2}} \le \sup_{\|u\|<\rho} \frac{1}{\left(\frac{1}{c_0}|v|_h - |u|_h\right)^{Q-2}} \le \frac{1}{\left(\frac{c'}{c_0}\|v\| - c''\rho^{\frac{1}{r}}\right)^{Q-2}} .$$

From this, (4.8) is easily obtained and the lemma is proved.

Now we can complete the proof of the Proposition (4.1). By (4.5) and (4.7) we can write, for every $R > 0$,

$$\begin{aligned} & E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)} \int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt \right) \\ . \qquad & \le c\,\mu_T^{(\varepsilon,x)}(B(0,R)) + \sup_{\|v\|\ge R} \int_{B(0,\rho)} G^{(\varepsilon,x)}(v,u)du \end{aligned} \tag{4.10}$$

(with the convention that $G(z,y) = 0$ if z or $y \notin \Omega$).

We can make small the second term in (4.10) by choosing a large R, as follows from (4.5) and (4.8). Hence, to finish the proof of (4.2), it suffices to prove the following:

(4.11) LEMMA. *For every $R > 0$,*

$$(4.12) \qquad \lim_{\varepsilon\downarrow 0, T\uparrow\infty} \mu_T^{(\varepsilon,x)}(B(0,R)) = 0\,.$$

Proof. Noting the result of the Corollary (3.18), the conclusion is obtained as soon as we show that, for every $R > 0$,

$$(4.13) \qquad \lim_{\varepsilon\downarrow 0} \mu_T^{(\varepsilon,x)}(B(0,R)) = \mu_T^{(x)}(B(0,R))\,.$$

For this, we write

$$\left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\mathbb{1}_{B(0,R)}(v_T^{(\varepsilon,x)})\right) - E_0\mathbb{1}_{B(0,R)}(u_T^{(x)})\right| \leq$$

$$E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T_{\bar{\Omega}}^{\varepsilon})}\left|\mathbb{1}_{B(0,R)}(v_T^{(\varepsilon,x)}) - \mathbb{1}_{B(0,R)}(u_T^{(x)})\right|\right) + P(T\geq\tau_\varepsilon) + 2\,P(T\geq T_{\bar{\Omega}}^{\varepsilon})\,.$$

As in the proof of the Proposition (2.12), it suffices to study the first term. But, the result of the Lemma (2.18) allows us to control this term, using the fact that u_T^x does not charge the boundary of the ball, and (4.13) follows.

This also ends the proof of the Proposition (4.1).

5. Proof of the Theorem (1.9)

To prove the Theorem (1.9) we need the following important:

(5.1) PROPOSITION. *Let H be a compact subset of $\mathbb{R}^d\setminus\{0\}$. Then*

$$(5.2) \qquad \lim_{\varepsilon\downarrow 0}\sup_{u\in H} |G^{(\varepsilon,x)}(0,u) - g^{(x)}(0,u)| = 0\,.$$

Proof. We shall show that, for $\varepsilon\downarrow 0$,

$$(5.3) \qquad G^{(\varepsilon,x)}(0,u)du \to g^{(x)}(0,u)du\,,\ \text{vaguely}\,,$$

and then, that there exists $\varepsilon_0 > 0$, such that $\{G^{(\varepsilon,x)}(0,\cdot), \varepsilon\in(0,\varepsilon_0]\}$ is a relatively compact subset of the set of continuous functions on H.

For the proof of (5.3), we denote $\mathrm{Lip}_\rho(\mathbb{R}^d)$ the set of all bounded Lipschitz continuous functions f on $\mathbb{R}^d$, with support in $B(0,\rho)$, such that $\|f\|_{\mathrm{Lip}}\leq 1$.

By (4.3) and (3.8), for every $f\in\mathrm{Lip}_\rho(\mathbb{R}^d)$,

$$\left|\int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} G^{(\varepsilon,x)}(0,u)f(u)du - \int_{\mathbb{R}^d} g^{(x)}(0,u)f(u)du\right|$$

$$\leq \left|E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_0^T f(v_t^{(\varepsilon,x)})dt\right) - E_0\int_0^T f(u_t^{(x)})dt\right|$$

$$+\left|E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt\right)\right| + T\,P(T\geq\tau_\varepsilon) + \left|E_0\int_T^\infty f(u_t^{(x)})dt\right|$$
$$\leq c\,T\,\varepsilon + \left|E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt\right)\right| + c\,T\,e^{-\frac{c'}{\varepsilon^2 T}} + \left|E_0\int_T^\infty f(u_t^{(x)})dt\right|\,,$$

as follows from (2.13) and from the classical exponential inequality. We can make small the last term by choosing a large T, as in (3.17). To control the second term we use (4.2). Doing so we get (5.3).

Now, we shall show that there exists $\varepsilon_0 > 0$, such that the functions $G^{(\varepsilon,x)}(0,\cdot)$, $\varepsilon\in(0,\varepsilon_0]$, are uniformly equicontinuous, provided they are restricted to the compact set H.

We prove the existence of a constant $c>0$, such that, for every $u\in H$, and $\varepsilon\in(0,\varepsilon_0]$,

$$|X^{J_j}G^{(\varepsilon,x)}(0,u)|\leq c\,,\ j=1,\dots,d\,. \tag{5.4}$$

By (4.4), for $u\in(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)$, we have

$$X^{J_j}G^{(\varepsilon,x)}(0,u)=\varepsilon^{Q-2}X^{J_j}G(x,u_\varepsilon^x)=\varepsilon^{Q-2+l_j}(X^{J_j}G)(x,u_\varepsilon^x)\,.$$

To obtain (5.4), we use another important estimate. It is similar to (1.7), but on the derivatives of G (see §6):

$$|X_{i_1}\dots X_{i_q}G(x,y)|\leq\frac{c}{|y|_x^{Q-2+q}}\,,\ y\neq x\,\text{close enough}\,. \tag{5.5}$$

Hence, for $j=1,\dots,d$,

$$\varepsilon^{Q-2+l_j}(X^{J_j}G)(x,u_\varepsilon^x)\leq\frac{c\,\varepsilon^{Q-2+l_j}}{|u_\varepsilon^x|_x^{Q-2+l_j}}=\frac{c}{|u|_h^{Q-2+l_j}}\,,$$

which is bounded when u lies in a compact set, and (5.4) is verified.

Using the weak convergence in (5.3) and the relatively compacteness of $\{G^{(\varepsilon,x)}(0,\cdot),\varepsilon\in(0,\varepsilon_0]\}$ on H, we can identifie the limit of $G^{(\varepsilon,x)}(0,\cdot)$. This ends the proof of (5.2).

Proof of the Theorem (1.9). We take

$$H=\{u\in\mathbb{R}^d:\sup(|u_j|:j=1,\dots,d)=1\}$$

and

$$\varepsilon_y=\sup(|u_j|^{\frac{1}{l_j}}:j=1,\dots,d)\,,$$

with $y\in\Omega$, $y=\varphi_x(u)$. Clearly,

$$\left(T_{\frac{1}{\varepsilon_y}}\circ\varphi_x^{-1}\right)(y)\in\left(T_{\frac{1}{\varepsilon_y}}\circ\varphi_x^{-1}\right)(\Omega)\cap H\,.$$

For every $\delta > 0$ and for every y sufficiently close to x, there exists $\varepsilon(\delta) > 0$, such that $\varepsilon_y \leq \varepsilon(\delta)$ and, by (5.1),

$$\left| G^{(\varepsilon,x)}\left(0, \left(T_{\frac{1}{\varepsilon_y}} \circ \varphi_x^{-1}\right)(y)\right) - g^{(x)}\left(0, \left(T_{\frac{1}{\varepsilon_y}} \circ \varphi_x^{-1}\right)(y)\right)\right| \leq \delta\,.$$

We note that, $u^{(x)}_{\varepsilon^2 t}$ and $T_\varepsilon(u^{(x)}_t)$ have the same law. Hence, by (3.8), we get

$$g^{(x)}\left(0, T_{\frac{1}{\varepsilon}}(u)\right) = \varepsilon^{Q-2} g^{(x)}(0,u)\,. \tag{5.6}$$

Then, using (4.4) and (5.6), for every $\delta > 0$ and for every y sufficiently close to x, $y = \varphi_x(u)$,

$$\left|\varepsilon_y^{Q-2} G(x,y) - \varepsilon_y^{Q-2} g^{(x)}(0,u)\right| \leq \delta\,. \tag{5.7}$$

Moreover, we can replace here ε_y by $|y|_x$ because, there exists $c > 0$ such that,

$$|y|_x \leq c\,\varepsilon_y\,. \tag{5.8}$$

Finally, let us denote, for $\theta \in \mathbb{R}^d \setminus \{0\}$,

$$\Phi_x(\theta) = g^{(x)}(0,\theta)\,. \tag{5.9}$$

As a consequence of the Proposition (3.9), Φ_x is a strictly positive smooth function on $\mathbb{R}^d \setminus \{0\}$.

By (1.8), for $y \neq x$,

$$\theta_x(y) = \left(T_{\frac{1}{|y|_x}} \circ \varphi_x^{-1}\right)(y)\,.$$

So, we conclude that, for every $\delta > 0$ and for every y sufficiently close to x,

$$\left||y|_x^{Q-2} G(x,y) - \Phi_x(\theta_x(y))\right| \leq \delta\,,$$

that is, (1.10).

The proof of the Theorem (1.9) is complete, except for the proof of Lemmas (A.1) and (A.7) of the Appendix and of the estimates (1.7), (3.13) and (5.5), which are simple consequences of [19] estimates, as we show in the following section.

6. Locally homogeneous norm associated to L

In this section we shall study the locally homogeneous norm $|\cdot|_x$ and we shall then justify the estimates (1.7), (3.13) and (5.5). It suffices to prove the following:

(6.1) PROPOSITION. *There exists some positive constants c, c', such that, for $y \neq x$ close enough,*

$$
(6.2) \qquad |G(x,y)| \leq \frac{c\,|y|_x^2}{m(B_h(x,|y|_x))}\,, \; |X_{i_1}\dots X_{i_q} G(x,y)| \leq \frac{c'\,|y|_x^{2-q}}{m(B_h(x,|y|_x))}\,.
$$

The estimates are then obtained using the simple calculation of the volume of a small homogeneous ball, $B_h(x,\varepsilon) = \{y : |y|_x < \varepsilon\}$:

$$
(6.3) \qquad m(B_h(x,\varepsilon)) = \int_{|y|_x<\varepsilon} dy = c\,\varepsilon^Q \int_{|v|_h<1} dv = c'\,\varepsilon^Q\,.
$$

Here we performed the change of variables $v = (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(y)$ and c' denotes a positive constant.

Proof of the Proposition (6.1). Noting the result of the Corollary in [19], p. 117, it is enough to show that there exists a positive constant c, such that, for y sufficiently close to x,

$$
(6.4) \qquad \rho(x,y) \leq c\,|y|_x\,.
$$

Recall that $\rho(x,y)$ is the distance introduced by [19], p. 107.

But by the Theorem 3 in [19], p. 112, ρ is locally equivalent to the pseudo-distance ρ_3. So, there exists a positive constant c, such that, for y sufficiently close to x,

$$
(6.5) \qquad \rho(x,y) \leq c\,\rho_3(x,y)\,.
$$

Recall that,

$$
\rho_3(x,y) = \inf\{\delta > 0 : \exists f \in C_3(\delta), f(0) = x, f(1) = y\}\,.
$$

Here $C_3(\delta) = \cup_D C_3(\delta, D)$, where, for each d-tuple D of multi-indices J, with $|J| \leq r$, $C_3(\delta, D)$ denote the class of smooth curves $f : [0,1] \to \mathbb{R}^d$, such that

$$
f'(t) = \sum_{J\in D} c_J X^J(f(t))\,, \text{ with } |c_J| < \delta^{|J|}\,,\; J \in D\,.
$$

We shall introduce a slight modification of the pseudo-distance ρ_3. We denote by $\mathcal{C}(\delta, B)$ the set of C^1-functions $f : [0,1] \to \mathbb{R}^d$, such that

$$
f'(t) = \sum_{j=1,\dots,d} c_j X^{J_j}(f(t))\,, \text{ with } \sum_{k=1}^{r} \left(\sum_{j,l_j=k} c_j^2 \right)^{\frac{Q}{2k}} < \delta^Q\,.
$$

Then we define,

$$d_B(x,y) = \inf\{\delta > 0 : \exists f \in \mathcal{C}(\delta, B), f(0) = x, f(1) = y\} \wedge 1 .$$

But

$$\sum_{k=1}^{r} \left(\sum_{j,l_j=k} c_j^2 \right)^{\frac{Q}{2k}} < \delta^Q \ \Rightarrow\ |c_j| < \delta^{l_j} \ , \ j = 1, \ldots, d \,,$$

so, $\mathcal{C}(\delta, B) \subset C_3(\delta)$. It follows that, for y sufficiently close to x,

$$\rho_3(x,y) \leq d_B(x,y) \,. \tag{6.6}$$

Moreover, by the definitions of $|y|_x$ and of $d_B(x,y)$, and by our assumptions on Ω, it is a simple observation that, for $x, y \in \Omega$,

$$d_B(x,y) = |y|_x \,. \tag{6.7}$$

This ends the proof of (6.4) and of the proposition.

(6.8) REMARK. Clearly, $d_B(x,y)$ is a pseudo-distance in the sense of [19], p. 109. From this, by (6.7), we see that there exists a constant $c_0 \geq 1$, such that, for every $x, y, z \in \Omega$,

$$|y|_x \leq c_0 \left(|z|_x + |z|_y \right) . \tag{6.9}$$

(6.10) REMARK. We can check another simple property of $|\cdot|_x$. For every $x, y \in \Omega$, $y = \varphi_x(u)$, there exists two positive constants, c', c'', such that

$$c' \, \|u\| \leq |y|_x \leq c'' \, \|u\|^{\frac{1}{r}} \,. \tag{6.11}$$

7. Capacity of small compact sets

In this section we shall estimate the capacity (relative to the kernel G) of small compact sets.

To apply the theory of Blumenthal and Getoor [6] for Markov processes in duality, we must consider the process (x_t) killed at an independent exponential random time ξ, of parameter $\lambda > 0$, which we denote by $(x_t^{(\lambda)})$.

The Green function of $(x_t^{(\lambda)})$ is the λ-potential of (x_t):

$$G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t^\Omega(x,y) dt \,. \tag{7.1}$$

(7.2) REMARK. The result of the Theorem (1.9) still holds with G replaced by G_λ. Indeed, we have

$$\left|G_\lambda(x,y)\,|y|_x^{Q-2} - \Phi_x(\theta_x(y))\right|$$

$$\leq \left|\frac{G_\lambda(x,y)}{G(x,y)} - 1\right| \cdot \left|G(x,y)\,|y|_x^{Q-2}\right| + \left|G(x,y)\,|y|_x^{Q-2} - \Phi_x(\theta_x(y))\right| .$$

The conclusion follows as soon as we show that,

$$\lim_{\varepsilon\downarrow 0} \sup_{\|y-x\|<\varepsilon} \left|\frac{G_\lambda(x,y)}{G(x,y)} - 1\right| = 0\,,$$

which can be done as in [9], p. 241.

Therefore, for every $\eta > 0$ and for every $y \neq x$ close enough,

$$\frac{-\eta + \Phi_x(\theta_x(y))}{|y|_x^{Q-2}} \leq G_\lambda(x,y) \leq \frac{\eta + \Phi_x(\theta_x(y))}{|y|_x^{Q-2}}\,. \tag{7.3}$$

Now, let us recall some definitions. By choosing $\lambda > 0$ large enough, we can apply the theory of [6] to the process $(x_t^{(\lambda)})$. For a compact subset H in Ω, we denote

$$T_H^{(\lambda)} = \inf\{t > 0 : x_t^{(\lambda)} \in H\}\,.$$

Let $\mu_H^{(\lambda)}$ the equilibrium measure of H, that is the unique finite measure supported by H such that, for every $x \in \Omega$,

$$P_x(T_H^{(\lambda)} < \infty) = G_\lambda \mu_H^{(\lambda)}(x) = \int_{\mathbb{R}^d} G_\lambda(x,y)\mu_H^{(\lambda)}(dy)\,. \tag{7.4}$$

The λ-capacity of H will be denoted by $c_\lambda(H)$, and is the total mass of $\mu_H^{(\lambda)}$, or, equivalently

$$c_\lambda(H) = \sup\{|\mu| : \mu \in \mathcal{M}(H), G_\lambda\mu \leq 1 \text{ on } \Omega\}\,. \tag{7.5}$$

Here $\mathcal{M}(H)$ is the set of all positive finite measures supported on H.

Let H be a compact subset of $\mathbb{R}^d$ containing 0. We shall describe the capacity of a small compact set. The natural dilation of H is $H_\varepsilon^x = (\varphi_x \circ T_\varepsilon)(H)$. We shall study the asymptotic behaviour of $c_\lambda\,(H_\varepsilon^x)$ as $\varepsilon \to 0$.

To write down the statement we need the following:

(7.6) LEMMA. *There exists*

$$\lim_{\varepsilon\downarrow 0} \frac{|v_\varepsilon^x|_{u_\varepsilon^x}}{\varepsilon} = \alpha(u,v) > 0 \tag{7.7}$$

and

(7.8) $$\lim_{\varepsilon\downarrow 0}\theta_{u^x_\varepsilon}(v^x_\varepsilon)=\beta(u,v)\neq 0\,,$$

for $u\neq v\in \mathbb{R}^d\setminus\{0\}$.

Proof. We have to calculate $|v^x_\varepsilon|_{u^x_\varepsilon}$. Since $\{X^{J_j}(y)\,:\,j=1,\ldots,d\}$ is a basis for y close to x, we have,

$$v^x_\varepsilon=\exp(Z_\varepsilon)(x)=\exp(W_\varepsilon)(u^x_\varepsilon)\,,\;u^x_\varepsilon=\exp(Y_\varepsilon)(x)\,,$$

with,

$$Z_\varepsilon=\sum_{j=1,\ldots,d}\varepsilon^{l_j}v_jX^{J_j}\,,\;Y_\varepsilon=\sum_{j=1,\ldots,d}\varepsilon^{l_j}u_jX^{J_j}\,,\;W_\varepsilon=\sum_{j=1,\ldots,d}w^\varepsilon_jX^{J_j}\,.$$

By the Campbell-Hausdorff formula we get,

$$Z_\varepsilon=W_\varepsilon+Y_\varepsilon+\frac{1}{2}[W_\varepsilon,Y_\varepsilon]+\ldots\,,$$

so,

$$w^\varepsilon_j=\varepsilon^{l_j}b_j(u,v)+O(\varepsilon^{l_j+1})\,,\;j=1,\ldots,d\,,\;b_j(u,v)\neq 0\,.$$

Using (1.5), we get

(7.9) $$|v^x_\varepsilon|_{u^x_\varepsilon}=\varepsilon\,\alpha(u,v)+O(\varepsilon^{1+\delta})\,,\;\delta\in(0,1)\,.$$

with,

(7.10) $$\alpha(u,v)=\left[\sum_{k=1}^r\left(\sum_{j,l_j=k}b_j(u,v)^2\right)^{\frac{Q}{2k}}\right]^{\frac{1}{Q}}\,.$$

This proves (7.7).

On the other hand, by (1.8) and the preceding calculation, we can write,

$$\theta_{u^x_\varepsilon}(v^x_\varepsilon)=\left(\frac{w^\varepsilon_j}{|v^x_\varepsilon|^{|J|}_{u^x_\varepsilon}}\right)_{j=1,\ldots,d}=\left(\frac{b_j(u,v)+O(\varepsilon)}{\alpha(u,v)^{l_j}+O(\varepsilon^\delta)}\right)_{j=1,\ldots,d}\,,\;\delta\in(0,1)\,.$$

Taking,

(7.11) $$\beta(u,v)=\left(\frac{b_j(u,v)}{\alpha(u,v)^{l_j}}\right)_{j=1,\ldots,d}$$

we get (7.8) and the lemma is proved.

We denote

$$r_x(u,v) = \frac{\Phi_x(\beta(u,v))}{\alpha(u,v)^{Q(x)-2}}\,,\; q_x(H) = \frac{m(H)}{\max_{u\in\partial H}\int_H r_x(u,v)dv}\,. \tag{7.12}$$

We can state now the main result of this section:

(7.13) PROPOSITION. *Let H be the closure of a bounded domain in* $\mathbb{R}^d$ *containing* 0, *and* $x \in \Omega$. *Then*

$$\lim_{\varepsilon\downarrow 0} \frac{c_\lambda\,(H^x_\varepsilon)}{\varepsilon^{Q(x)-2}} = q_x(H)\,. \tag{7.14}$$

Proof. We consider ν, the measure with the density $\mathbb{1}_H$ with respect to the Lebesgue measure and ν^x_ε, the image measure of ν through $\varphi_x \circ T_\varepsilon$.

A lower bound for $c_\lambda\,(H^x_\varepsilon)$ is obtained as soon as we can obtain a uniform bound on $G_\lambda \nu^x_\varepsilon$. By the maximum principle of Bony [7], for hypoelliptic operators, it suffices to bound $G_\lambda\nu^x_\varepsilon$ on H^x_ε.

Take $u^x_\varepsilon \in H^x_\varepsilon$. Then,

$$G_\lambda\nu^x_\varepsilon(u^x_\varepsilon) = \int_{\mathbb{R}^d} G_\lambda(u^x_\varepsilon, v)\nu^x_\varepsilon(dv) = \int_H G_\lambda(u^x_\varepsilon, v^x_\varepsilon)dv\,.$$

Then, by (7.3) and (7.9),

$$G_\lambda\nu^x_\varepsilon(u^x_\varepsilon) \le \int_H \frac{\eta + \Phi_{u^x_\varepsilon}(\theta_{u^x_\varepsilon}(v^x_\varepsilon))}{|v^x_\varepsilon|^{Q-2}_{u^x_\varepsilon}}dv = \int_H \frac{\eta + \Phi_{u^x_\varepsilon}(\theta_{u^x_\varepsilon}(v^x_\varepsilon))}{\varepsilon^{Q-2}(\alpha(u,v)^{Q-2} + O(\varepsilon^\delta))}dv\,.$$

Using (7.5), for all $u \in H$,

$$\frac{c_\lambda\,(H^x_\varepsilon)}{\varepsilon^{Q-2}} \ge \frac{m(H)}{\int_H \frac{\eta+\Phi_{u^x_\varepsilon}(\theta_{u^x_\varepsilon}(v^x_\varepsilon))}{\alpha(u,v)^{Q-2}+O(\varepsilon^\delta)}dv}\,.$$

Hence, by the continuity of $\Phi_x(\theta)$ and by (7.11), we get,

$$\liminf_{\varepsilon\downarrow 0} \frac{c_\lambda\,(H^x_\varepsilon)}{\varepsilon^{Q-2}} \ge q_x(H)\,. \tag{7.15}$$

On the other hand,

$$\nu^x_\varepsilon G_\lambda(u^x_\varepsilon) = \int_{\mathbb{R}^d} G_\lambda(v, v^x_\varepsilon)\nu^x_\varepsilon(dv) = \int_H G_\lambda(v^x_\varepsilon, u^x_\varepsilon)dv = \int_H G_\lambda(u^x_\varepsilon, v^x_\varepsilon)dv\,,$$

so, again by (7.3) and (7.9),

$$\nu^x_\varepsilon G_\lambda(u^x_\varepsilon) \ge \int_H \frac{-\eta + \Phi_{u^x_\varepsilon}(\theta_{u^x_\varepsilon}(v^x_\varepsilon))}{|v^x_\varepsilon|^{Q-2}_{u^x_\varepsilon}}dz = \int_H \frac{-\eta + \Phi_{u^x_\varepsilon}(\theta_{u^x_\varepsilon}(v^x_\varepsilon))}{\varepsilon^{Q-2}(\alpha(u,v)^{Q-2} + O(\varepsilon^\delta))}dv\,.$$

We denote by $\mu^{\lambda,x}_{\varepsilon,H}$, the equilibrium measure of H^x_ε. We can write,

$$|\mu^{\lambda,x}_{\varepsilon,H}| \cdot \int_H \frac{-\eta + \Phi_{u^x_\varepsilon}(\theta_{u^x_\varepsilon}(v^x_\varepsilon))}{\varepsilon^{Q-2}(\alpha(u,v)^{Q-2} + O(\varepsilon^\delta))} dv \le \int_{\mathbb{R}^d} \mu^{\lambda,x}_{\varepsilon,H}(du^x_\varepsilon)\nu^x_\varepsilon G_\lambda(u^x_\varepsilon)$$

$$= \int_{\mathbb{R}^d} \nu^x_\varepsilon(dv) G_\lambda \mu^{\lambda,x}_{\varepsilon,H}(v) \le |\nu^x_\varepsilon| = m(H)\,.$$

Hence, for all $u \in H$,

$$\frac{c_\lambda\,(H^x_\varepsilon)}{\varepsilon^{Q-2}} \cdot \int_H \frac{-\eta + \Phi_{u^x_\varepsilon}(\theta_{u^x_\varepsilon}(v^x_\varepsilon))}{\alpha(u,v)^{Q-2} + O(\varepsilon^\delta)} dv \le m(H)\,,$$

from which we get, by (7.11),

$$\limsup_{\varepsilon\downarrow 0} \frac{c_\lambda\,(H^x_\varepsilon)}{\varepsilon^{Q-2}} \le q_x(H)\,. \tag{7.16}$$

8. Applications: various sample path properties

As was said in [9], p. 222, as soon as we dispose of the results on the Green function and on the capacity of small compact sets, we can derive some sample path properties. The general methods used in [9], §7 and §8, can be applied.

We note that, for certain properties we do not need the exact behaviour of G, but only the estimates

$$\frac{c'}{|y|_x^{Q-2}} \le G(x,y) \le \frac{c}{|y|_x^{Q-2}}\,,\ \text{with}\, x \ne y\, \text{close enough}\,, \tag{8.1}$$

c, c' being positive constants. The right hand is (1.7) and the left hand can be obtained in a similar way as (1.7), that is, using the estimate on the volume of homogeneous small balls, (6.3) and the Theorem I (ii) in [10], p. 248.

We shall emphasize only the differences with respect to the case considered by [9].

(a) Hitting probabilities of small compact sets.

For $\varepsilon > 0$ sufficiently small, we denote,

$$T_{H^x_\varepsilon} = \inf\{t > 0 : x_t \in H^x_\varepsilon\}\,.$$

(8.2) PROPOSITION. *For $n \ge 1$ integer, for $x_0, x_1, \ldots, x_n$ distinct points of Ω and for $t \ge 0$,*

$$\lim_{\varepsilon\downarrow 0} \left(\frac{1}{\varepsilon^{Q(x)-2}}\right)^n P_{x_0}(T_{H^{x_1}_\varepsilon} \le \ldots \le T_{H^{x_n}_\varepsilon} \le t) = q_{x_1}(H) \ldots q_{x_n}(H) \tag{8.3}$$

$$\cdot \qquad \times \int_{0\le s_1\le \ldots \le s_n\le t} ds_1 \ldots ds_n \, p^\Omega_{s_1}(x_0,x_1) p^\Omega_{s_2-s_1}(x_1,x_2)\ldots p^\Omega_{s_n-s_{n-1}}(x_{n-1},x_n)\,.$$

Moreover, there exists constants $c, c_{n,t} > 0$, independent of $x_0, x_1, \ldots, x_n$, such that, whenever $|x_j|_{x_{j-1}} \ge c\,\varepsilon$, $j = 1,\ldots,n$

$$\left(\frac{1}{\varepsilon^{Q(x)-2}}\right)^n P_{x_0}(T_{H^{x_1}_\varepsilon} \le \ldots \le T_{H^{x_n}_\varepsilon} \le t) \le c_{n,t} \prod_{j=1}^{n} \frac{1}{|x_j|^{Q(x)-2}_{x_{j-1}}}\,. \tag{8.4}$$

For the proof we use the result on the capacity (7.11) and we repeat the arguments in pp. 250-252, [9].

(b) Wiener sausage.

We shall analyse the asymptotic behaviour of the volume of the Wiener sausage of small radius. For $0 \le t < \tau$, let us denote

$$\mathcal{S}_{H^x_\varepsilon}(0,t) = \bigcup_{0\le s\le t} H^{x_s}_\varepsilon\,, \tag{8.5}$$

the "sausage" associated to (x_t) and to H^x_ε, $H \subset \mathbb{R}^d$, containing 0.

By a similar proof as in [9], pp. 253-257, we could obtain:

(8.6) PROPOSITION. *Let $\mu(dx) = f(x)\,dx$, where f is a bounded measurable function on Ω. Then, for every $p \ge 1$, $0 < T < \tau$, $x_0 \in \Omega$,*

$$\lim_{\varepsilon\downarrow 0} E_{x_0}\left[\sup_{0\le t\le T}\left|\frac{1}{\varepsilon^{Q(x)-2}}\mu\left(\mathcal{S}_{H^x_\varepsilon}(0,t)\right) - \int_0^t f(x_s) q_{x_s}(H) ds\right|^p\right] = 0\,. \tag{8.7}$$

(8.8) REMARK. Recall that $(\mathcal{G}_t)$ denote the invariant diffusion on $\mathcal{N}(m,r)$. Let us denote, for $\varepsilon > 0$, $t \ge 0$,

$$\mathcal{S}^N_\varepsilon(0,t) = \{g \in \mathcal{N}(m,r) : |g\cdot \mathcal{G}_s^{-1}|_N \le \varepsilon\,, \text{ for some } s \le t\}\,. \tag{8.9}$$

If μ denotes the Haar measure on the group, by the Theorem (4.9) in [12], we get,

$$\lim_{t\uparrow\infty} \frac{1}{t}\,\mu(\mathcal{S}^N_1(0,t)) = c\,,\ P_e - \text{a.s.} \tag{8.10}$$

From this we obtain a similar result as (7.q) in [9], p. 258:

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^{Q_N-2}}\,\mu(\mathcal{S}^N_\varepsilon(0,1)) = c, \ \ \text{in probability}\,. \tag{8.11}$$

Indeed, if δ_ε denotes the image on $\mathcal{N}(m,r)$ of the dilation on the algebra $g(m,r)$

(see [2], p. 88), then $(\delta_\varepsilon(\mathcal{G}_s))$ and $(\mathcal{G}_{\varepsilon^2 s})$ have the same law. By scaling and homogeneity properties we can show that $\mu(\mathcal{S}_\varepsilon^N(0,1))$ and $\varepsilon^{Q_N}\mu(\mathcal{S}_1^N(0,\frac{1}{\varepsilon^2}))$ have the same law.

(c) Double points.

We could prove the same result as the Theorem 8.2 in [9], p. 261:

(8.12) PROPOSITION. *For every* $x \in \Omega$, with P_x *probability one, the process* $\{x_s : 0 \le s < \tau\}$ *does not have double points.*

In proving this, we use the Hausdorff measure with respect to the homogeneous norm $|\cdot|_x$ and the estimates for G, (8.1). The difference with respect to [9] is that, instead (8.d), p. 262, we prove:

$$E_x\left(\sup_{s<\delta\wedge\tau} |x_s|_x^{Q(x)-2}\right) \le c\,\delta^{\frac{Q(x)-2}{2}}\,,$$

for every $x \in \Omega$, $\delta \in (0,1)$, c being a positive constant. For this, we use the Taylor stochastic expansion and the fact that, for every multi-index J, there exists a constant $c(J) > 0$, such that, $E(|B_t^J|^2) \le c(J)\,t^{|J|}$ (see [1], p. 34).

(d) Wiener and Poincaré tests.

The result which we formulate is similar to the classical Wiener test. For another form we refer to [5], p. 98.

Let us consider a constant α greather than the constant $c_0 \ge 1$, which appears in the triangular inequality for the homogeneous norm $|\cdot|_x$, (6.9). For B a Borel set contained in U we denote

$$B_n = \{y \in B : \frac{1}{\alpha^{n+1}} \le |y|_x \le \frac{1}{\alpha^n}\}\,,\; n \ge 1\,. \tag{8.13}$$

(8.14) PROPOSITION. *The probability* $P_x(T_B^{(\lambda)} = 0) = 0$ *or* 1 *according as the series* $\sum_n \alpha^{n(Q(x)-2)} c_\lambda(B_n)$ *converges or diverges.*

We show that, for $n \ge 1$,

$$c'\,\alpha^{n(Q-2)} c_\lambda(B_n) \le P_x(T_{B_n}^{(\lambda)} < \infty) \le c\,\alpha^{(n+1)(Q-2)} c_\lambda(B_n)\,,$$

using the estimates in (8.1). Then we conclude as in [12], pp. 108-110.

This result could be applied to obtain the cone test of Poincaré. A homogeneous cone with vertex 0 is a Borel set C with non-empty interior, which is stable for the dilations T_α and such that $0 \in \partial C$.

(8.15) COROLLARY. *Consider C a homogeneous cone with vertex 0 and N a neighbourhood of 0. If B is a Borel set such that $\varphi_x(N \cap C) \subset B \subset U$, then $P_x(T_B^{(\lambda)} = 0) = 1$.*

We note that $C_{n+1} = T_{1/\alpha}(C_n)$, $n \geq 1$, so, by a simple property of the capacity (see [12] Proposition (4.7)), we get, $c_\lambda(C_n) = c\alpha^{-n(Q-2)}$, $c > 0$. Then we can conclude, using the Proposition (8.14), since $\varphi_x((N \cap C)_n) \subset B_n$ and $c_\lambda(\varphi_x(N \cap C)) = c\, c_\lambda(N \cap C)$, $c > 0$, (see also the Corollary (5.4) [12]).

9. Examples

In this section we shall describe some concrete examples, where we can perform more calculations. Firstly, let us point out some simple cases.

We consider on $\mathbb{R}^3$ the vector fields $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$, $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$. Then $[X_1, X_2] = -4\,\partial_{x_3}$ and the operator $L = \frac{1}{2}(X_1^2 + X_2^2)$ is hypoelliptic. This case is called the Heisenberg case and in [11], p. 375 (see also [13], p. 101) was calculated the Green function on $\mathbb{R}^3$ with pole 0:

$$G^H(0, y) = \frac{1/(4\pi)}{\sqrt{(y_1^2 + y_2^2)^2 + y_3^2}^{\,4-2}} = \frac{1/(4\pi)}{|y|_0^{Q_H - 2}}\,. \tag{9.1}$$

In [9] a more general situation is treated. Consider two smooth vector fields X_1, X_2 on $\mathbb{R}^3$, such that for every $x \in \Omega$, $X_1(x)$, $X_2(x)$, $[X_1, X_2](x)$ span $\mathbb{R}^3$. Then the Green function satisfies:

$$\big|G(x, y)\, d(x, y)^{4-2} - c\big| \to 0\,, \text{ as}\, y \to x\,. \tag{9.2}$$

It is also shown that the pseudo-distance $d(x, y)$ is equivalent to $|y|_x$.

We firstly treat the following:

(a) Curved Heisenberg case.

For $n \geq 1$ integer, we take $m = 2n$ and $d = 2n+1$. Suppose that $X_1, \ldots, X_{2n}$ are smooth vector fields on $\mathbb{R}^{2n+1}$, such that,

$$[X_{2k-1}, X_{2k}] = [X_1, X_2]\,,\; k = 1, \ldots, n\,, \tag{9.3}$$

all other brackets being zero. Let us consider Ω a bounded domain in $\mathbb{R}^{2n+1}$. We shall suppose that, for every $x \in \Omega$, the vectors $X_1(x), \ldots, X_{2n}(x)$, $[X_1, X_2](x)$ span $\mathbb{R}^{2n+1}$.

It is a particular case because we consider only two order brackets and a single one is not zero. In this case $r = 2$ and $Q = 2n + 2$. The basis is indexed by $B = \{1, 2, \ldots, 2n, (1, 2)\}$.

The diffusion associated to the vector fields, starting from a fixed point

$x \in \Omega$, is

$$(9.4)\quad x_t = \exp\left(\sum_{j=1}^{2n} B_t^j X_j - \frac{1}{2}\sum_{k=1}^{n}\int_0^t (B_s^{2k}dB_s^{2k-1} - B_s^{2k-1}dB_s^{2k})[X_1, X_2]\right)(x)$$
$$. \qquad +t^{\frac{3}{2}} R_3(t)\,,$$

as we can see by (2.14). We must compare (x_t) to the left invariant diffusion on the Heisenberg group, H_{2n+1}, with its usual structure on $\mathbb{R}^{2n+1}$. The left invariant vector fields are defined by

$$Y_{2k-1} = \partial_{x_{2k-1}} - 2\,x_{2k}\,\partial_{x_{2n+1}}\,,\ Y_{2k} = \partial_{x_{2k}} + 2\,x_{2k-1}\,\partial_{x_{2n+1}}\,,\ k = 1, \ldots, n\,,$$

so, the invariant diffusion started from 0 is

$$(9.5)\quad \mathcal{G}_t = \exp\left(\sum_{j=1}^{2n} B_t^j Y_j - \frac{1}{2}\sum_{k=1}^{n}\int_0^t (B_s^{2k}dB_s^{2k-1} - B_s^{2k-1}dB_s^{2k})[Y_1, Y_2]\right)(0)\,.$$

In this case we do not need any projection, and $(u_t^{(x)})$ is the diffusion

$$(9.6)\qquad \left(B_t^1, \ldots, B_t^{2n}, -\frac{1}{2}\sum_{k=1}^{n}\int_0^t (B_s^{2k}dB_s^{2k-1} - B_s^{2k-1}dB_s^{2k})\right)\,.$$

Its Green fonction, $g^{(x)}$, is the invariant Green function on the Heisenberg group. By the result of [11], p. 375, we get

$$(9.7)\qquad g^{(x)}(0, y) = \frac{1/c_n}{\left[\left(\sum_{j=1}^{2n} y_j^2\right)^2 + y_{2n+1}^2\right]^{\frac{n}{2}}}\,,\ c_n = \frac{2^{n-1}\Gamma(\frac{n}{2})}{\pi^{n+1}}\,.$$

For $y = \varphi_x(y_1, \ldots, y_{2n+1})$, we denote

$$(9.8)\qquad |y|_x = \left[\left(\sum_{j=1}^{2n} y_j^2\right)^{n+1} + |y_{2n+1}|^{n+1}\right]^{\frac{1}{2n+2}}\,.$$

Then, applying the Theorem (1.9), we obtain

$$(9.9)\qquad \lim_{\varepsilon\downarrow 0}\ \sup_{\|x-y\|<\varepsilon} \left|G(x, y)\,|y|_x^{2n} - \Phi_x(\theta_x(y))\right| = 0\,.$$

Here,

$$(9.10)\qquad \theta_x(y) = \left(\frac{y_1}{|y|_x}, \ldots, \frac{y_{2n}}{|y|_x}, \frac{y_{2n+1}}{|y|_x^2}\right)$$

and

$$\Phi_x(t_1,\dots,t_{2n+1}) = \frac{c_x/c_n}{\left[\left(\sum_{j=1}^{2n} t_j^2\right)^2 + t_{2n+1}^2\right]^{\frac{n}{2}}}\,,\ c_x > 0\,. \tag{9.11}$$

(9.12) REMARK. Noting the symetry of the first $2n$ coordinates, we can write a simpler form of (9.9). Put

$$\vartheta(y) = \frac{|y_{2n+1}|}{\sum_{j=1}^{2n} y_j^2}\,,\ \Psi(t) = \frac{1}{c_n}\,\frac{(1+t^{n+1})^{\frac{n}{n+1}}}{(1+t^2)^{\frac{n}{2}}}\,. \tag{9.13}$$

Then, by (9.9), we get

$$\lim_{\varepsilon\downarrow 0}\ \sup_{\|y-x\|<\varepsilon} \left|G(x,y)\,|y|_x^{2n} - c_x\,\Psi(\vartheta(y))\right| = 0\,. \tag{9.14}$$

We also note that, for $n=1$, $\Psi = \frac{1}{c_n}$ is constant and we can compare (9.14) with the result obtained by [9], (9.2).

(9.15) REMARK. In this particular case we could easily write the result on the capacity of small compact sets.

Now, we shall study a slight extension of the last model. Let us replace (9.3) by the following assumption:

$$[X_{2k-1}, X_{2k}] = a_k[X_1, X_2]\,,\ a_k \in \mathbb{R}^*\,,\ k=1,\dots,n\,, \tag{9.16}$$

all other hypothesis on the vector fields being the same.

The associated diffusion can be written as in (9.4), using the Taylor stochastic expansion. It will be compared to the diffusion $(\tilde{u}_t^x)$ generated by the following vector fields:

$$Y_{2k-1} = \partial_{x_{2k-1}} + 2\,a_k\,x_{2k}\,\partial_{x_{2n+1}}\,,\ Y_{2k} = \partial_{x_{2k}} - 2\,a_k\,x_{2k-1}\,\partial_{x_{2n+1}}\,,\ k=1,\dots,n\,,$$

that is,

$$\left(B_t^1,\dots,B_t^{2n}, -\frac{1}{2}\sum_{k=1}^n a_k \int_0^t (B_s^{2k}dB_s^{2k-1} - B_s^{2k-1}dB_s^{2k})\right)\,. \tag{9.17}$$

The Green function associated to $(\tilde{u}_t^x)$ was pointed out by [16], p. 136:

$$\tilde{g}^{(x)}(0,y) = c_n \int_{\mathbb{R}} \frac{A(s)\,ds}{\left(\sum_{k=1}^n b_k(s)(y_{2k-1}^2 + y_{2k}^2) + i\,s\,y_{2n+1}\right)^n}\,, \tag{9.18}$$

where $i=\sqrt{-1}$, $c_n = \frac{(n-1)!}{2\pi}$ and

$$(9.19) \qquad A(s) = \frac{1}{(4\pi)^n} \prod_{k=1}^{n} \frac{4a_k s}{\sinh(4a_k s)} \,, \; b_k(s) = (a_k s)\coth(4a_k s)\,.$$

(9.20) REMARK. When $(y_1, \dots, y_{2n+1}) = (0, \dots, 0, y_{2n+1})$, with $y_{2n+1} \neq 0$, we must integrate in (9.19) on $\mathbb{R} + i\,q$, $q > 0$ (see also [4]).

We can obtain the behaviour of the Green function $\tilde{G}$, associated to the vector fields X_j, as in the first case. We use the same homogeneous norm, given by (9.8), and we get the same relation as (9.9), with Φ_x replaced by:

$$(9.21) \qquad \tilde{\Phi}_x(t_1, \dots, t_{2n+1}) = c_x\, c_n \int_{\mathbb{R}} \frac{A(s)\,ds}{\left(\sum_{k=1}^{n} b_k(s)(t_{2k-1}^2 + t_{2k}^2) + i\,s\,t_{2n+1}\right)^n}\,.$$

(9.22) REMARK. We can simplify the result again, using the symetry of the pairs of coordinates.

(9.23) REMARK. We can find again the result of [9], for $n = 1$. Also, we could formulate the result on the capacity.

(9.24) REMARK. A more general situation can be obtained assuming that $m = 2n$, $d = 2n + p$ (p missing directions, $p \geq 1$, integer) and $r = 2$. Using some recent results of [4] we could write similar results.

As was said, we shall describe a case when the condition that the geometry of the brackets is locally constant fails:

(b) A case at step larger than two.

Let us consider on $\mathbb{R}^3$ the vector fields

$$(9.25) \quad X_1 = \partial_{x_1} + 2\,p\,x_2(x_1^2 + x_2^2)^{p-1}\partial_{x_3}\,, \; X_2 = \partial_{x_2} - 2\,p\,x_1(x_1^2 + x_2^2)^{p-1}\partial_{x_3}\,,$$

with $p \geq 1$, integer, and $L = \frac{1}{2}(X_1^2 + X_2^2)$. The case $p = 1$ is the classical Heisenberg case $H_3 = \mathcal{N}(2,2)$.

The operator L is nowhere elliptic, but is hypoelliptic. Indeed, for $p > 1$ and for $x \notin \{x_1 = x_2 = 0\}$, we have

$$[X_1, X_2] = -8\,p\,(x_1^2 + x_2^2)^{p-1}\partial_{x_3}\,.$$

So, for $p > 1$ and for $x \notin \{x_1 = x_2 = 0\}$, $X_1(x), X_2(x)$ and $[X_1, X_2](x)$ span $\mathbb{R}^3$. This situation was already treated. On the other hand we see that for the points on the axis $\{x_1 = x_2 = 0\}$, to span $\mathbb{R}^3$ we need to go up to the brackets of order $2\,p$ in this points. This time $r(0,0,x_3) = 2\,p$ and $Q(0,0,x_3) = 2\,p+2$. Clearly, the geometry of the brackets is not locally constant around the point $(0,0,x_3)$.

Operators like L occur in the study of the boundary of the Cauchy-Riemann complex (see [17]). Precisely, let us consider the domain

$$\mathcal{D} = \{(z_1, z) \in \mathbb{C}^2 \,,\, \operatorname{Im} z_1 > |z|^{2p}\} \,.$$

If $p = 1$, $\mathcal{D}$ is the generalized upper half plane in $\mathbb{C}^2$. The vector field $\partial_z - 2\,i\,\bar{z}\,\partial_{z_1}$ is the unique holomorphic vector field which is tangent to the boundary $b\mathcal{D}$ of $\mathcal{D}$. In the tangential coordinate system (see [17]: coordinates $\rho = \operatorname{Im} z_1 - |z|^2$, $z, \bar{z}$ and $x_3 = \operatorname{Re} z_1$) this vector field takes the form

$$Z = \partial_z + i\,\bar{z}\,\partial_{x_3} \,.$$

Z is left-invariant with respect to the nilpotent group structure, the Heisenberg group, on $\mathbb{R}^3 = b\mathcal{D}$.

In the case $p > 1$, we have

$$Z = \partial_z + i\,p\,z^{p-1}\bar{z}^p\,\partial_{x_3}$$

and there is no group structure on $\mathbb{R}^3$ with respect to which Z is left-invariant.

We also note that $Z = \frac{1}{2}\,X_1 - \frac{i}{2}\,X_2$ and L is of the type $-\Box_b$, precisely,

$$L = Z\,\bar{Z} + \bar{Z}\,Z \,.$$

Recall that in the Heisenberg case, the Green function on $\mathbb{R}^3$ is known. By left-invariance it suffices to know the Green function with pole $(0,0,0)$ (see (9.1)).

In [15] the case $p = 2$ is considered and the expression of the Green function on $\mathbb{R}^3$ with arbitrary pole is given.

Here we consider an arbitrary p. As was said, the case when the pole is outside of the axis $\{x_1 = x_2 = 0\}$ was treated. It is plausible that the method of [15] can give an exact formula for the Green function with arbitrary pole. However, the calculation seems to be more delicate (see also [17], p. 157). Nevertheless, we can give an exact formula for the Green function with pole on the axis $\{x_1 = x_2 = 0\}$:

(9.26) PROPOSITION. *The Green function on* $\mathbb{R}^3$, *associated to the vector fields* X_1, X_2, *with pole* $(0,0,x_3)$, *is*

$$G((0,0,x_3),(y_1,y_2,y_3)) = \frac{1/(4\,p\,\pi)}{\sqrt{(y_1^2+y_2^2)^{2p} + (y_3-x_3)^2}} \,. \tag{9.27}$$

Proof. We denote $w = y_1 + i\,y_2$, $\sigma^2 = |w|^{4p} + (y_3 - x_3)^2$ and we must show that the Green function is

$$G((0,x_3),(w,y_3)) = \frac{1}{4\,p\,\pi\,\sigma} \,.$$

Clearly, this function is a C^∞-function of (w, y_3), as long as $(w, y_3) \neq (0, x_3)$.

We consider, for $\epsilon > 0$, the C^∞-function on $\mathbb{R}^3$,

$$G_\epsilon((0,x_3),(w,y_3)) = \frac{1}{4\,p\,\pi\,\sigma_\epsilon}\,,$$

where $\sigma_\epsilon^2 = (|w|^{2p} + \epsilon^{2p})^2 + (y_3 - x_3)^2$.

Then, $G_\epsilon((0,x_3),(w,y_3)) \to G((0,x_3),(w,y_3))$, pointwise as $\epsilon \downarrow 0$, as long as $(w, y_3) \neq (0, x_3)$. In fact, we can show that

$$G((0,x_3),(w,y_3)) = \lim_{\epsilon \downarrow 0} G_\epsilon((0,x_3),(w,y_3))\,, \text{ as a distribution in } \mathbb{R}^3\,.$$

Indeed, we see that there exists a positive constant c, independent of ϵ, such that $|G_\epsilon| \leq \frac{c}{\sigma}$. If we show that $\frac{1}{\sigma}$ is locally integrable, then, by the Lebesgue dominated convergence theorem we get

$$G_\epsilon((0,x_3),\cdot) \to G((0,x_3),\cdot)\,, \text{ in } \mathcal{D}'(\mathbb{R}^3)\,, \text{ as } \epsilon \downarrow 0\,.$$

We study the integrability at $(w, y_3) = (0, x_3)$ and we may suppose that $x_3 = 0$. We shall estime $\frac{1}{\sigma}$ on the domain $|w| \leq 1$, $|y_3| \leq 1$. We have,

$$\int_{-1}^{1} \frac{dy_3}{\sigma} = 2\,\log\left[1 + (1 + |w|^{4p})^{1/2}\right] - 4\,p\,\log|w|\,.$$

The first term is clearly integrable on $|w| \leq 1$, as for the second, $\int_{|w|\leq 1} |\log|w||$ $\times dv(w) = 2\,\pi \int_0^1 r\,\log r dr < \infty$.

After some calculations, we get

$$L\left(\frac{1}{4\,p\,\pi\,\sigma_\epsilon}\right) = \frac{p}{2\,\pi} \cdot \frac{\epsilon^{2p}\,|w|^{2p-2}}{\sigma_\epsilon^3}\,.$$

Hence, we have,

$$LG = 0\,, \text{ as long as } (w, y_3) \neq (0, x_3)$$

and

$$LG_\epsilon((0,x_3),(w,y_3)) \to 0\,, \text{ as } \epsilon \downarrow 0\,,$$

uniformly on compact subsets of $\mathbb{R}^3$ which do not contain the point $(0, x_3)$.

We show that

$$\int_{\mathbb{R}^3} LG_\epsilon((0,x_3),(w,y_3))dv(w,y_3) = 1\,.$$

Indeed,

$$\frac{p}{2\,\pi}\epsilon^{2p}\,|w|^{2p-2} \int_{\mathbb{R}} \frac{dy_3}{\left[(|w|^{2p} + \epsilon^{2p})^2 + (y_3 - x_3)^2\right]^{3/2}} = \frac{p}{\pi} \cdot \frac{\epsilon^{2p}\,|w|^{2p-2}}{(|w|^{2p} + \epsilon^{2p})^2}$$

and then,

$$\frac{p}{\pi}\,\epsilon^{2p}\int_{\mathbb{R}^2}\frac{|w|^{2p-2}\,dv(w)}{(|w|^{2p}+\epsilon^{2p})^2}=2\,p\,\epsilon^{2p}\int_0^\infty\frac{r^{2p-1}\,dr}{(r^{2p}+\epsilon^{2p})^2}=1\,.$$

Now we consider an arbitrary $\phi\in C_0^\infty(\mathbb{R}^3)$. Then, for any neighbourhood U of $(0,x_3)$, we can write,

$$< G((0,x_3),\cdot), L\,\phi >=\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^3} G_\epsilon((0,x_3),(w,y_3))L\,\phi(w,y_3)dv(w,y_3)$$

$$=\lim_{\epsilon\downarrow 0}\phi(0,x_3)\int_{\mathbb{R}^3} L\,G_\epsilon((0,x_3),(w,y_3))dv(w,y_3)$$

$$+\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^3} L\,G_\epsilon((0,x_3),(w,y_3))(\phi(w,y_3)-\phi(0,x_3))dv(w,y_3)$$

$$=\phi(0,x_3)+\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^3\setminus U} L\,G_\epsilon((0,x_3),(w,y_3))(\phi(w,y_3)-\phi(0,x_3))dv(w,y_3)$$

$$+\lim_{\epsilon\downarrow 0}\int_{U} L\,G_\epsilon((0,x_3),(w,y_3))(\phi(w,y_3)-\phi(0,x_3))dv(w,y_3)=\phi(0,x_3)\,.$$

This proves the fact that G is the Green function of L on $\mathbb{R}^3$ with pole $(0,x_3)$.

(9.28) REMARK. In the Heisenberg case, the Green function with arbitrary pole is given by (9.27). For the case treated in [15], $p=2$, the Green function with arbitrary pole has two terms, the first being the right hand of (9.27). In the general case we should attempt to find p terms for the Green function with arbitrary pole, the first being the right hand of (9.30).

The diffusion started from $(0,0,0)\in\{x_1=x_2=0\}$, generated by X_1,X_2 is

$$x_t=\left(B_t^1,B_t^2,4\,p\int_0^t R_s^{2(p-1)}dS_s\right)\,, \tag{9.29}$$

where

$$R_t^2=(B_t^1)^2+(B_t^2)^2\,,\;S_t=\frac{1}{2}\int_0^t B_s^2dB_s^1-B_s^1dB_s^2\,. \tag{9.30}$$

We denote, for $y=\varphi_{(0,0,0)}(y_1,y_2,y_3)$,

$$|y|_0=\left[(y_1^2+y_2^2)^{p+1}+|y_3|^{\frac{p+1}{p}}\right]^{\frac{1}{2p+2}}\,, \tag{9.31}$$

$$\theta_0(y)=\left(\frac{y_1}{|y|_0},\frac{y_2}{|y|_0},\frac{y_3}{|y|_0^{2p}}\right)\,, \tag{9.32}$$

and

$$\Phi_0(t_1,t_2,t_3) = \frac{1}{4\,p\,\pi}\cdot\frac{1}{\sqrt{(t_1^2+t_2^2)^{2p}+t_3^2}}\,. \tag{9.33}$$

Then, by (9.27),

$$G(0,y) = \frac{\Phi_0(\theta_0^1(y),\theta_0^2(y),\theta_0^3(y))}{|y|_0^{2p}}\,. \tag{9.34}$$

Clearly, we could use the symmetry of the first two coordinates to write another expression for the Green function (see [14]).

Finally, we shall consider the:

(c) Grushin case.

Let us consider on $\mathbb{R}^2$ the vector fields

$$X_1 = \partial_{x_1}\,,\ X_2 = x_1\partial_{x_2}\,. \tag{9.35}$$

Then $[X_1,X_2] = \partial_{x_2}$ and the operator $L = \frac{1}{2}(X_1^2+X_2^2)$ is hypoelliptic on the axis $\{x_1=0\}$ and elliptic elsewhere.

We consider the point $x=(0,0)$, which lies on the axis $\{x_1=0\}$. Clearly, $r(0,0)=2$, $Q(0,0)=3$ and $B=\{1,(12)\}$.

The diffusion started from x is

$$x_t = \left(B_t^1, \int_0^t B_s^1 dB_s^2\right) = \left(B_t^1, \frac{B_t^1 B_t^2}{2} - S_t\right)\,, \tag{9.36}$$

where S_t is as in (9.30).

The left invariant diffusion started from 0 on the Heisenberg group H_3 is

$$\mathcal{G}_t = (B_t^1, B_t^2, -S_t)\,. \tag{9.37}$$

Therefore,

$$x_t = \pi_x(\mathcal{G}_t)\,,\ \pi_x(a,b,c) = (a, \frac{a\,b}{2}+c)\,. \tag{9.38}$$

From this it is not difficult to see that the Green function of (x_t) is

$$G((0,0),(y_1,y_2)) = \int_{\mathbb{R}} G^H\left((0,0,0),(y_1,h,y_2-\frac{y_1\,h}{2})\right)dh\,, \tag{9.39}$$

or, by (9.1)

$$G((0,0),(y_1,y_2)) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{dh}{\sqrt{(y_1^2+h^2)^2+(y_2-\frac{y_1 h}{2})^2}}\,. \tag{9.40}$$

Let us denote

$$|(y_1,y_2)|_0 = \sqrt[3]{|y_1|^3+|y_2|^{\frac{3}{2}}}\,. \tag{9.41}$$

If we take

$$\theta_0(y_1,y_2) = \left(\frac{y_1}{|(y_1,y_2)|_0}, \frac{y_2}{|(y_1,y_2)|_0^2}\right) \tag{9.42}$$

and

$$\Phi_0(t_1,t_2) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{dh}{\sqrt{(t_1^2+h^2)^2+(t_2-\frac{t_1 h}{2})^2}}\,, \tag{9.43}$$

then

$$G((0,0),(y_1,y_2)) = \frac{\Phi_0(\theta_0(y_1,y_2))}{|(y_1,y_2)|_0^{3-2}}\,. \tag{9.44}$$

(9.45) REMARK. We could take as angular variable $\vartheta_0(y_1,y_2) = \frac{y_2}{y_1^2}$ to write another expression for the Green function (see [14]).

(9.46) REMARK. In this case several of our hypothesis fail: $d = 2$, $Q(x) = 3$, the geometry of the brackets is not locally constant in x and the estimates of [19] are not proved. Nevertheless, the result obtained by a direct calculation, (9.44) is quite close to the result of the Theorem (1.9).

Appendix

We prove here the integral estimates which we used in the proof of the Theorem (1.9).

We shall denote $d_k = \mathrm{card}\{j : l_j = k\}$, $k = 1,\dots,r$. So, $d = \sum_{k=1}^r d_k$ and $Q = \sum_{k=1}^r k d_k$. We assume that $r \geq 2$, $d_1 \geq 2$ and $d_k \geq 1$, $k = 2,\dots,r$.

(A.1) LEMMA. *There exists two positive constants c_0, c_1, such that, for every $S > 0$,*

$$\mathcal{I} = \int_{|u|_h<S} \frac{du}{|u|_h^{Q-2}} < c\,S^{\frac{2}{r^2}}\,, \tag{A.2}$$

where $c = c_0(2\pi)^{l-r}c_1^{r-1}$ except for $r = 2$, $d_2 = 1$ where $c = \sqrt{2}(2\pi)^2$.

Proof. In estimating $\mathcal{I}$ we shall use the following simple observation. Let us denote, for $n \geq 1$, $p, q > 0$ and $\sigma > 0$,

$$\Lambda_{n,p,q}(\sigma) = \int_0^\sigma \frac{\rho^{n-1}}{(\rho^q + 1)^p} d\rho \,. \tag{A.3}$$

Clearly, $\Lambda_{n,p,q}$ is increasing and we see that, there exists $c_1 > 0$, depending only on n, p, q, such that

$$\lim_{\sigma \uparrow \infty} \Lambda_{n,p,q}(t) < c_1 \text{ , provided } pq - n > 0 \,. \tag{A.4}$$

Also, for $S, R > 0$, we have

$$\int_0^S \frac{\rho^{n-1}}{(\rho^q + R)^p} d\rho = R^{\frac{n-pq}{q}} \Lambda_{n,p,q}\left(\frac{S}{R^{\frac{1}{q}}}\right) . \tag{A.5}$$

We shall denote, for $k = 1, \ldots, r$,

$$s_k^2 = \sum_{j, l_j = k} u_j^2 \, , \; Q_k = \sum_{i=k}^r i \, d_i \,. \tag{A.6}$$

Then

$$\begin{aligned}
\mathcal{I} &\leq \int_{\{u : |s_k| < S^{\frac{1}{k}}, k=1,\ldots,r\}} \frac{du}{\left(\sum_{k=1}^r s_k^{\frac{Q}{k}}\right)^{\frac{Q-2}{Q}}} \\
&= \int_{\{|s_k| < S^{\frac{1}{k}}, k=2,\ldots,r\}} du'' \int_{\{|s_1| < S\}} \frac{du'}{(s_1^Q + R_1)^{\frac{Q_1 - 2}{Q}}} \,,
\end{aligned}$$

where $du' = \prod_{j, l_j = 1} du_j$, $du'' = \prod_{j, 2 \leq l_j \leq r} du_j$ and $R_1 = \sum_{k=2}^r s_k^{\frac{Q}{k}}$. By a simple change of variables and by (A.5), we get

$$\begin{aligned}
\int_{\{|s_1| < S\}} \frac{du'}{(s_1^Q + R_1)^{\frac{Q_1-2}{Q}}} &= (2\pi)^{d_1 - 1} \int_0^S \frac{\rho^{d_1 - 1} d\rho}{(\rho^Q + R_1)^{\frac{Q_1 - 2}{Q}}} \\
&= (2\pi)^{d_1 - 1} R_1^{\frac{d_1 - Q_1 + 2}{Q}} \Lambda_{d_1, \frac{Q_1 - 2}{Q}, Q}\left(\frac{S}{R_1^{\frac{1}{Q}}}\right) .
\end{aligned}$$

We have $Q \cdot \frac{Q_1 - 2}{Q} - d_1 = Q_2 - 2$.

The case $r = 2$, $d_2 = 1$ will be considered separately. For $r = 2$ and $d_2 > 1$, by (A.4) we get

$$\mathcal{I} < c \int_{\{|s_2| < S^{\frac{1}{2}}\}} \frac{du''}{s_2^{\frac{Q}{2} \cdot \frac{Q_2 - 2}{Q}}} = c \int_0^{S^{\frac{1}{2}}} \frac{\rho^{d_2 - 1} d\rho}{\rho^{\frac{2\, d_2 - 2}{2}}} = c\, S^{\frac{1}{2}} \, , \; c = (2\pi)^{d_1 + d_2 - 2} c_1 \,.$$

For $r \geq 3$, again by (A.4), we can write

$$\mathcal{I} < c \int_{\{|s_k| < S^{\frac{1}{k}}, k=3,\ldots,r\}} du'' \int_{\{|s_2| < S^{\frac{1}{2}}\}} \frac{du'}{(s_2^{\frac{Q}{2}} + R_2)^{\frac{Q_2-2}{Q}}} ,$$

where, this time $du' = \prod_{j,l_j=2} du_j$, $du'' = \prod_{j,3 \leq l_j \leq r} du_j$ and $R_2 = \sum_{k=3}^r s_k^{\frac{Q}{k}}$. By a similar calculation:

$$\int_{\{|s_2| < S^{\frac{1}{2}}\}} \frac{du'}{(s_2^{\frac{Q}{2}} + R_2)^{\frac{Q_2-2}{Q}}} = (2\,\pi)^{d_2-1} \int_0^{S^{\frac{1}{2}}} \frac{\rho^{d_2-1} d\rho}{(\rho^{\frac{Q}{2}} + R_2)^{\frac{Q_2-2}{Q}}}$$

$$= (2\,\pi)^{d_2-1} R_2^{\frac{2\,d_2 - Q_2 + 2}{Q}} \Lambda_{d_2, \frac{Q_2-2}{Q}, \frac{Q}{2}} \left(\frac{S^{\frac{1}{2}}}{R_2^{2/Q}} \right) .$$

Since $Q \cdot \frac{Q_2-2}{Q} - 2\,d_2 = Q_3 - 2 > 0$, we get

$$\mathcal{I} < c \int_{\{|s_k| < S^{\frac{1}{k}}, k=3,\ldots,r\}} R_2^{-\frac{Q_3-2}{Q}} du'' \,,\ c = (2\pi)^{d_1+d_2-2} c_1^2 \,.$$

For $r = 3$, $d_3 = 1$, we have $\mathcal{I} < c\,S^{\frac{2}{9}}$, with $c = 3(2\pi)^{d_1+d_2-2} c_1^2$, and for $r = 3$, $d_3 > 1$,

$$\mathcal{I} < c \int_{\{|s_3| < S^{\frac{1}{3}}\}} \frac{du''}{s_3^{\frac{Q}{3} \cdot \frac{Q_3-2}{Q}}} = (2\pi)^{d_3-1} c \int_0^{S^{\frac{1}{3}}} \frac{\rho^{d_3-1} d\rho}{\rho^{\frac{3\,d_3-2}{3}}} = c\,S^{\frac{2}{9}} ,$$

with $c = \frac{3}{2}(2\pi)^{d_1+d_2+d_3-3} c_1^2$.

For $r \geq 4$ we repeat the reasoning and (A.2) is obtained in a finite number of steps.

To finish the proof we must treat the case $r = 2$, $d_2 = 1$. We have

$$\mathcal{I} \leq \int_{\{|s_1| < S, |s_2| < S^{\frac{1}{2}}\}} \frac{du_1 du_2 du_3}{\sqrt{s_1^4 + s_2^2}} = 2\pi \int_{(0,S) \times (0, S^{\frac{1}{2}})} \frac{\rho\, d\rho\, dz}{\sqrt{\rho^4 + z^2}} \leq (2\pi)^2 \sqrt{2S} \,.$$

This ends the proof of (A.2).

Before stating the second result of this section we introduce some notations. Recall that $n = \dim g(m,r) - d = \operatorname{card} A - \operatorname{card} B$. Put $A \setminus B = \{L_1, \ldots, L_n\}$, $m_i = |L_i|$, $i = 1, \ldots, n$ and $e_k = \operatorname{card}\{i : m_i = k\}$, $k = 1, \ldots, r$. So, $n = \sum_{k=1}^r e_k$ and $Q_N = \sum_{k=1}^r k(d_k + e_k)$. For a point $(u,h) \in \mathbb{R}^d \times \mathbb{R}^n$ we denote

$$|(u,h)|_N = \left[\sum_{k=1}^r \left(\sum_{j,l_j=k} u_j^2 + \sum_{i,m_i=k} h_i^2 \right)^{\frac{Q_N}{2k}} \right]^{\frac{1}{Q_N}} .$$

(A.7) LEMMA. *For every $S > 0$, there exists a positive constant c, such that*

$$\int_{\{|u|_h < S\}\times \mathbb{R}^n} \frac{du\, dh}{|(u,h)|_N^{Q_N-2}} < c \tag{A.8}$$

Proof. Let us denote, for $k = 1, \dots, r$

$$t_k^2 = \sum_{i, m_i = k} h_i^2\,,\; Q_{N,k} = \sum_{i=k}^{r} i\,(d_i + e_i)\,. \tag{A.9}$$

Replacing in (A.2), d by $d+n$ and $|u|_h$ by $|(u,h)|_N$, we get the existence of a constant $c > 0$, such that for every $U > 0$,

$$\int_{|(u,h)|_N < U} \frac{du\, dh}{|(u,h)|_N^{Q_N-2}} < c\, U^{\frac{2}{r^2}}\,. \tag{A.10}$$

So, it suffices to prove that, for every $S, T > 0$, there exists a constant $c > 0$, such that

$$\mathcal{J} = \int_{\{|s_k| < S^{\frac{1}{k}}, |t_k| \geq T^{\frac{1}{k}}, k=1,\dots,r\}} \frac{du\, dh}{\left[\sum_{k=1}^{r} (s_k^2 + t_k^2)^{\frac{Q_N}{2k}}\right]^{\frac{Q_N-2}{Q_N}}} < c \tag{A.11}$$

We see that, for $S, T > 0$, $b \geq 1$ and for $a \geq 2$ and $p \geq 2$ or $a = 1$ and $p \geq 3$, there exists a constant $c_2 > 0$, such that

$$\int_0^S ds \int_T^\infty dt\, \frac{s^{a-1} t^{b-1}}{(s^2 + t^2)^{\frac{p(a+b)-2}{2p}}} \leq c_2\,. \tag{A.12}$$

Indeed, we have to study only the integral in t and, clearly,

$$\frac{t^{b-1}}{(s^2+t^2)^{\frac{p(a+b)-2}{2p}}} \sim \frac{1}{t^{1+a-\frac{2}{p}}}\,,\ \text{as}\, t \uparrow \infty\,.$$

We proceed as in the proof of the Lemma (A.1):

$$\mathcal{J} = \int_{\{|s_k| < S^{\frac{1}{k}}, |t_k| \geq T^{\frac{1}{k}}, k=2,\dots,r\}} du''\, dh'' \int_{|s_1| < S, |t_1| \geq T} \frac{du'\, dh'}{\left[(s_1^2 + t_1^2)^{\frac{Q_N}{2}} + R_1\right]^{\frac{Q_{N,1}-2}{Q_N}}}\,,$$

where $du' = \prod_{j, l_j = 1} du_j$, $dh' = \prod_{i, m_i = 1} h_i$, $du'' = \prod_{j, 2 \leq l_j \leq r} du_j$, $dh'' = \prod_{i, 2 \leq m_i \leq r} dh_i$, $R_1 = \sum_{k=2}^{r} (s_k^2 + t_k^2)^{\frac{Q_N}{2k}}$. Using again (A.5) and (A.4), we get

$$\begin{aligned} \mathcal{J} &< c \int_{\{|s_k| < S^{\frac{1}{k}}, |t_k| \geq T^{\frac{1}{k}}, k=2,\dots,r\}} du''\, dh'' \int_0^\infty \frac{\rho^{d_1 + e_1 - 1} d\rho}{(\rho^{Q_N} + R_1)^{\frac{Q_{N,1}-2}{Q_N}}} \\ &< c\, c_1 \int_{\{|s_k| < S^{\frac{1}{k}}, |t_k| \geq T^{\frac{1}{k}}, k=2,\dots,r\}} du''\, dh''\, R_1^{\frac{d_1 + e_1 - Q_{N,1} + 2}{Q_N}}\,. \end{aligned}$$

Here we used the fact that $Q_N \cdot \frac{Q_{N,1}-2}{Q_N} - (d_1+e_1) = Q_{N,2}-2 > 0$, excepting the case when $r=2$, $d_2=1$ and $e_2=0$ which will be treated separately.

For $r=2$, $d_2>1$ and $e_2\geq 1$, we can write, by (A.12),

$$\int_{|s_2|<S^{\frac12},|t_2|\geq T^{\frac12}} \frac{du''\,dh''}{(s_2^2+t_2^2)^{\frac{Q_N}{4}\cdot\frac{Q_{N,2}-2}{Q_N}}} = \int_0^{S^{\frac12}} ds \int_{T^{\frac12}}^{\infty} dt \frac{s^{d_2-1}t^{e_2-1}}{(s^2+t^2)^{\frac{2\,(d_2+e_2)-2}{4}}} < c_2\,.$$

For $r\geq 3$ we repeat the reasoning:

$$\mathcal{J} < c\,c_1 \int_{\{|s_k|<S^{\frac1k},|t_k|\geq T^{\frac1k},k=3,\ldots,r\}} du''\,dh''$$
$$\times \int_{|s_2|<S^{\frac12},|t_2|\geq T^{\frac12}} \frac{du'\,dh'}{\left[(s_2^2+t_2^2)^{\frac{Q_N}{4}}+R_2\right]^{\frac{Q_{N,2}-2}{Q_N}}}\,,$$

where $du'=\prod_{j,l_j=2} du_j$, $dh'=\prod_{i,m_i=2} h_i$, $du''=\prod_{j,3\leq l_j\leq r} du_j$, $dh''=\prod_{i,3\leq m_i\leq r} dh_i$, $R_2=\sum_{k=3}^r (s_k^2+t_k^2)^{\frac{Q_N}{2k}}$. Then, by (A.5) and (A.4), we get

$$\mathcal{J} < c\,c_1 \int_{\{|s_k|<S^{\frac1k},|t_k|\geq T^{\frac1k},k=3,\ldots,r\}} du''\,dh'' \int_0^\infty \frac{\rho^{d_2+e_2-1}d\rho}{(\rho^{\frac{Q_N}{2}}+R_2)^{\frac{Q_{N,2}-2}{Q_N}}}$$
$$< c\,c_1^2 \int_{\{|s_k|<S^{\frac1k},|t_k|\geq T^{\frac1k},k=3,\ldots,r\}} du''\,dh''\,R_2^{\frac{2(d_2+e_2)-Q_{N,2}+2}{Q_N}}\,,$$

since $\frac{Q_N}{2}\cdot\frac{Q_{N,2}-2}{Q_N} - (d_2+e_2) = Q_{N,3}-2>0$.

If $r=3$, we have, by (A.12),

$$\int_{|s_3|<S^{\frac13},|t_3|\geq T^{\frac13}} \frac{du''\,dh''}{(s_3^2+t_3^2)^{\frac{Q_N}{6}\cdot\frac{Q_{N,3}-2}{Q_N}}} = \int_0^{S^{\frac13}} ds \int_{T^{\frac13}}^{\infty} dt \frac{s^{d_3-1}t^{e_3-1}}{(s^2+t^2)^{\frac{3\,(d_3+e_3)-2}{6}}} < c_2\,.$$

For $r\geq 4$ we repeat the calculation and (A.11) is obtained in a finite number of steps.

Finally we treat the case $r=2$, $d_2=1$, $e_2=0$:

$$\mathcal{J} = c\int_0^S \int_T^\infty \int_0^{S^{\frac12}} \frac{s^{d_1-1}t^{e_1-1}ds\,dt\,dz}{[(s^2+t^2)^{\frac{Q_N}{2}}+z^{\frac{Q_N}{2}}]^{\frac{Q_N-2}{Q_N}}} < c\,c_2 S^{\frac12}\,.$$

This ends the proof of (A.8).

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